Introduction to Topology   Exercises I

Section A exercises are the easiest; Section B are to be handed in; Section C are harder. Exam-type questions are indicated with an asterisk. Some Section A questions may be helpful in Section B.
Hand-in time: 2pm Wednesday of week 2.

Section A

1. Find explicit homeomorphisms

1. $(0,1) \to \mathbb{R}$.
2. $(0,1) \to (0,\infty)$
3. $\mathbb{R} \times S^1 \to \mathbb{R}^2 \smallsetminus \{(0,0)\}$
4. $[0,1] \times S^1 \to \{(x,y) \in \mathbb{R} : 1 \leq \|(x,y)\| \leq 2\}$.

2. Show that the following pairs of spaces are *not* homeomorphic:

1. S^1 and $[0,1]$ (Hint: consider what happens if you remove a single point from each.)
2. The letter "X" (four line-segments sharing a common end-point) and the letter "Y" (three line segments sharing a common end-point).
3. S^2 and the torus (the surface of a doughnut), using the fact that every closed curve in S^2 separates it (i.e. the complement has at least two connected components).

4. Show that if X is a compact topological space and $\sim$ is an equivalence relation, then $X/\sim$ is also compact. Hint: $q : X \to X/\sim$ is continuous.

5. Suppose that $f : X \to Y$ is a continuous map. Define a relation $\sim$ on X by setting $x_1 \sim x_2$ if and only if $f(x_1) = f(x_2)$. Show
(i) $\sim$ is an equivalence relation.
(ii) If f is surjective then there is a continuous bijection $X/\sim \ \to Y$.

We call $\sim$ *the equivalence relation induced by* f.

Section B

Exercises 6, 7 and 8 use the *compact-to-Hausdorff Lemma* stated in Wednesday's lecture: If $f : X \to Y$ is a continuous bijection, and if X is compact and Y is Hausdorff, then f is a homeomorphism. This is Proposition 5.2 in the online Lecture Notes.

6. (i) Find a surjective continuous map from the closed interval $[-1,1]$ to the unit circle S^1 that is injective except that it sends 1 and -1 to the same point. You could define it using a formula, or by means of a good picture.
(ii) Define an equivalence relation on $[-1,1]$ by declaring the two end-points, 1 and -1, to be equivalent. Use (i) and Proposition 3.1 in the online Lecture Notes to find a continuous bijection $[-1,1]/\sim \ \to S^1$.

(iii) Show that $[-1, 1]/\sim$ is homeomorphic to S^1.

[6 marks]

7. Let D^2 be the closed unit disc in $\mathbb{R}^2$, and define an equivalence relation on D^2 by setting $x_1 \sim x_2$ if $||x_1|| = ||x_2|| = 1$. Show that $D^2/\sim$ is homeomorphic to S^2. Hint: first define a surjection $D^2 \to S^2$ mapping all of ∂D^2 to the north pole. You could define it using a formula, or by means of a good picture.

[6 marks]

8. Suppose that $f : X \to Y$ is continuous and surjective, and let $\sim$ be the equivalence relation induced by f (see Q5 above).

Show that if X is compact and Y is Hausdorff then $X/\sim$ is homeomorphic to Y.

Which theorem of group theory does this resemble?

[3 marks]

9. The *real projective plane* $\mathbb{RP}^2$ is defined as the quotient of the 2-sphere S^2 by the equivalence relation $x \sim -x$ which identifies antipodal points. Show that $\mathbb{RP}^2$ is Hausdorff.

[5 marks]

Section C

10. On the closed unit ball D^n in $\mathbb{R}^n$, define an equivalence relation by setting $x_1 \sim x_2$ if $||x_1|| = ||x_2|| = 1$. Show that $D^n/\sim$ is homeomorphic to S^n.

11. Suppose that X is a topological space with an equivalence relation $\sim$, and let Q be the quotient space. Suppose that $X_1 \subset X$ meets every equivalence class. The restriction of $\sim$ to X_1 determines an equivalence relation $\sim_1$ on X_1. Let Q_1 be the quotient. Suppose that Q_1 is compact and Q is Hausdorff. Complete the following diagram to show that Q_1 and Q are homeomorphic:

$$\begin{array}{ccc} X_1 & \hookrightarrow & X \\ {\scriptstyle q_1}\downarrow & & \downarrow{\scriptstyle q} \\ Q_1 & & Q \end{array}$$

12. (i) Let Q be the quotient of $\mathbb{R}^3 \setminus \{0\}$ by the equivalence relation

$$x_1 \sim x_2 \quad \text{if there exists } \lambda \in \mathbb{R} \text{ such that } \lambda x_1 = x_2.$$

Show, assuming that Q is Hausdorff, that the space $\mathbb{RP}^2$ as defined[1] in Exercise 9 is homeomorphic to Q. Hint: use Exercise 11.
(ii) Show that Q is Hausdorff.

13. Let Q be the quotient of the unit disc $D \subset \mathbb{R}^2$ by the equivalence relation

$$x_1 \sim x_1 \quad \text{if } ||x_1|| = ||x_2|| = 1 \text{ and } x_1 = -x_2.$$

Show that Q is homeomorphic to $\mathbb{RP}^2$. Hint: D is homeomorphic to a hemisphere in S^2.

[1] In fact the definition of $\mathbb{RP}^2$ given here in Exercise 12 is more useful and more usual.