Problem 3.1 We use Fourier series to show that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. By Corollary 3.57 of Rynne & Youngson, we know that the set

$$E = \{e_n(x) = (2\pi)^{-1/2} e^{inx} : n \in \mathbb{Z}\}$$

is an orthonormal basis for $L^2_{\mathbb{C}}[-\pi, \pi]$.

Setting $f(x) = x$ on $[-\pi, \pi]$, we have

$$\|f\|^2 = \int_{-\pi}^{\pi} x\overline{x} dx = \int_{-\pi}^{\pi} x^2 \, \mathrm{d}x = \frac{2}{3}\pi^3.$$

Now let $n \in \mathbb{Z} \setminus \{0\}$. We compute (uv)' = u'v + uv' int uv' = uv - int u'v

$$\begin{aligned}
(2\pi)^{1/2}(f, e_n) &= \int_{-\pi}^{\pi} x e^{inx} \, \mathrm{d}x \\
&= \frac{1}{in}(xe^{inx})|_{-\pi}^{\pi} - \frac{1}{in}\int_{-\pi}^{\pi} e^{inx} \, \mathrm{d}x \\
\int_{-\pi}^{\pi} e^{inx} \, \mathrm{d}x &= \frac{1}{in}e^{in\pi} - \frac{1}{in}e^{-in\pi} \\
&= \frac{2}{n}\frac{e^{in\pi} - e^{-in\pi}}{2i} = 2\sin(n\pi)/n = 0 \\
(xe^{inx})|_{-\pi}^{\pi} &= \pi e^{i\pi n} + \pi e^{-i\pi n} = 2\pi\cos(n\pi) = 2\pi(-1)^n \\
(2\pi)^{1/2}(f, e_n) &= \frac{2\pi(-1)^n}{in}.
\end{aligned}$$

Furthermore, we have

$$(2\pi)^{1/2}(f, e_0) = \int_{-\pi}^{\pi} x e^{0ix} \, \mathrm{d}x = \int_{-\pi}^{\pi} x \, \mathrm{d}x = 0$$

as $f(x) = x$ is an odd function on $[-\pi, \pi]$. Using Parseval's theorem, we obtain

$$\begin{aligned}
\frac{2}{3}\pi^3 = \|f\|^2 &= \sum_{n=-\infty}^{\infty} |(f, e_n)|^2 \\
&= \sum_{n=-\infty}^{-1} \left| \frac{2\pi(-1)^n}{(2\pi)^{1/2} in} \right|^2 + 0 + \sum_{n=1}^{\infty} \left| \frac{2\pi(-1)^n}{(2\pi)^{1/2} in} \right|^2 \\
&= 4\pi \sum_{n=1}^{\infty} \frac{1}{n^2}
\end{aligned}$$

so $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Problem 3.2 (a) Let X be a Banach space and suppose Y is a closed linear subspace of X. We show that X/Y is again a Banach space with norm $\|[x]\| = \inf_{u \in [x]} \|u\|$.

We first show that this newly defined norm is in fact a norm. Let $[x], [x'] \in X/Y$ and $\alpha \in \mathbb{F}$.

Clearly, we have $\|[0]\| = 0$, as $0 \in Y$. Suppose $\|[x]\| = 0$. Then $\inf_{u \in [x]} \|u\| = 0$, and there exists a sequence $\{u_n\} \subset [x]$ such that $\|u_n\| \to 0$ as $n \to \infty$. But then $u_n \to 0$, and as $[x] = x + Y$ is closed, $0 \in [x]$. Thus we have $[x] = [0]$, and $\|[x]\| = 0$ implies that $[x] = [0]$.

Furthermore, we have

$$\begin{aligned}
\|\alpha[x]\| = \|[\alpha x]\| &= \inf_{u \in [\alpha x]} \|u\| \\
&= \inf_{u \in [x]} \|\alpha u\| = |\alpha| \inf_{u \in [x]} \|u\| = \|[x]\|
\end{aligned}$$

and

$$\begin{aligned}\|[x]+[x']\| &= \inf_{u\in[x+x']}\|u\| = \inf_{u\in[x],v\in[x']}\|u+v\| \\ &\le \inf_{u\in[x]}\|u\| + \inf_{v\in[x']}\|v\| = \|[x]\| + \|[x']\|\,.\end{aligned}$$

As such, $\|\cdot\|$ is a norm.

We now show that X/Y is complete with respect to this norm. Let $\{[x_n]\}$ be a sequence in X/Y such that $\sum_{n=1}^{\infty}\|[x_n]\|$ converges. We show that $\sum_{n=1}^{\infty}[x_n]$ converges; then by Exercise 2.2 of Homework Assignment 2, we obtain that X/Y is complete.

Note that for each $n \in \mathbb{N}$, there must exist some $u_n \in [x_n]$ such that $\|u_n\| \le \|[x_n]\| + \frac{1}{2^n}$ as $\|[x_n]\| = \inf_{u\in[x_n]}\|u\|$. Then we have

$$\sum_{n=1}^{N}\|u_n\| \le \sum_{n=1}^{N}\|[x_n]\| + \frac{1}{2^n}$$

so $\sum_{n=1}^{\infty}\|u_n\| \le \sum_{n=1}^{\infty}\|[x_n]\| + 1 < \infty$ and $\sum_{n=1}^{\infty}\|u_n\|$ converges. As X is Banach, $\sum_{n=1}^{\infty} u_n$ must converge as well. Call the limit of this series u. Now note that the map $X \to X/Y$ given by $x \mapsto [x]$ is continuous, as $\|[x]\| = \inf_{u\in[x]}\|u\| \le \|x\|$. As such, we obtain

$$\sum_{n=1}^{\infty}[x_n] = \sum_{n=1}^{\infty}[u_n] = \lim_{N\to\infty}\sum_{n=1}^{N}[u_n] = \lim_{N\to\infty}\left[\sum_{n=1}^{N}u_n\right] = \left[\lim_{N\to\infty}\sum_{n=1}^{N}u_n\right] = [u]$$

so that $\sum_{n=1}^{\infty}[x_n]$ converges. Thus we conclude that X/Y is Banach.

(b) Let $\mathscr{H}$ be a Hilbert space and let Y be a closed linear subspace of $\mathscr{H}$.

Define $([x],[y]) := (\pi(x),\pi(y))$ where $\pi : \mathscr{H} \to Y^{\perp}$ denotes the projection of $\mathscr{H}$ onto $Y^{\perp}$. We define the projection π as follows. Let $x \in \mathscr{H}$. As Y is a closed linear subspace and thus convex and non-empty, there exists a unique $q \in Y$ such that

$$\|x-q\| = \inf\{\|x-y\| : y \in Y\}$$

Then define $\pi(x) = x - q$. Note that $\pi(x)$ is uniquely determined by the property that $(x-\pi(x)) \perp Y^{\perp}$.

Claim. *$\pi : \mathscr{H} \to Y^{\perp}$ is a linear map.*

Proof. Let $\alpha,\beta \in \mathbb{F}$ and $x,x' \in \mathscr{H}$. Then, for any $w \in Y^{\perp}$,

$$(\alpha x + \beta x' - \alpha\pi(x) - \beta\pi(x'), w) = \alpha(x-\pi(x), w) + \beta(x'-\pi(x'), w) = 0$$

as $(x-\pi(x)) \perp Y^{\perp}$ and $(x'-\pi(x')) \perp Y^{\perp}$. By uniqueness of the projection, we must have $\pi(\alpha x + \beta x') = \alpha\pi(x) + \beta\pi(x')$. □

Claim. *$(\cdot,\cdot)$ is an inner product.*

Proof. First observe that $(\cdot,\cdot)$ is well defined, as two different representatives of an equivalence class $[x] \in \mathscr{H}/Y$ differ by an element of Y, and Y is precisely the kernel of $\pi : \mathscr{H} \to Y^{\perp}$.

Let $[x],[y],[z] \in \mathscr{H}/Y$ and $\alpha,\beta \in \mathbb{F}$. We have $([0],[0]) = (\pi(0),\pi(0)) = (0,0) = 0$. Now suppose $([x],[x]) = 0$. Then $(\pi(x),\pi(x)) = 0$, so $\pi(x) = 0$, and $x \in \ker\pi = Y$. But then $[x] = 0$, by definition of the quotient space $\mathscr{H}/Y$.

We have

$$([x],[y]) = (\pi(x),\pi(y)) = \overline{(\pi(y),\pi(x))} = \overline{([x],[y])}$$

and

$$\begin{aligned}(\alpha[x]+\beta[y],[z]) &= (\pi(\alpha x+\beta y),\pi(z)) \\ &= \alpha(\pi(x),\pi(z))+\beta(\pi(y),\pi(z)) = \alpha([x],[z])+\beta([y],[z])\end{aligned}$$

so we conclude that $(\cdot,\cdot)$ is indeed an inner product. □

Claim. $\|[x]\| = \|\pi(x)\|$, *where* $\|[x]\|$ *is the norm defined in part (a).*

Proof.

$$\|[x]\| = \inf_{u\in[x]}\|u\| = \inf_{x-u\in Y}\|x-(x-u)\| = \|\pi(x)\|$$

by definition of the projection π. □

Then we have

$$([x],[x]) = (\pi(x),\pi(x)) = \|\pi(x)\|^2 = \|[x]\|^2$$

and thus the norm and inner product on $\mathscr{H}/Y$ are compatible. As such, we get completeness of $\mathscr{H}/Y$ with respect to the inner product by part (a).

Problem 3.3 Let $(X,\|\cdot\|)$ be a Banach space.

(a) For each $k\in\mathbb{N}$, let $A_k\subset X$ be compact and $r_k\in\mathbb{R}$, $r_k>0$, such that $A_{k+1}\subseteq A_k+\overline{B_{r_k}(0)}$, and $\sum_{k=1}^\infty r_k<\infty$. We show that $A := \overline{\bigcup_{k=1}^\infty A_k}$ is compact.
We use the following extension of the Heine-Borel theorem.

Theorem (Extension of Heine-Borel)**.** *Let (M,d) be a metric space. Then M is compact if and only if it is complete and totally bounded. Here, totally bounded means that for every $\epsilon>0$, there exist finitely many $x_1,\dots,x_n\in M$ such that $M=\cup_{k=1}^n B_\epsilon(x_k)$.*

Let $d(x,y):=\|x-y\|$ be the metric on X. Clearly, A is complete, as it is a closed subspace of the complete metric space X. We now show that A is totally bounded. Let $\epsilon>0$ be given. Set $\delta=\sum_{k=1}^\infty r_k+1$ and choose N such that $\sum_{k=1}^\infty r_k<\sum_{k=1}^{N-1} r_k+\frac{\epsilon}{3\delta}$. Since each of the $A_1,\dots,A_N$ is compact, there exist $x_1,\dots,x_L\in\bigcup_{j=1}^N A_j$ with $\bigcup_{j=1}^N A_j\subseteq\bigcup_{i=1}^L B_{\epsilon/\delta}(x_i)$. We now show that $A\subseteq\bigcup_{i=1}^L B_\epsilon(x_i)$. Let $y\in A$; then $d(y,\bigcup_{n=1}^\infty A_n)=0$, so there must exist some minimal M such that $d(y,A_M)<\epsilon/(3\delta)$.
If $M\le N$, then $y\in\bigcup_{j=1}^N A_j\subseteq\bigcup_{i=1}^L B_{\epsilon/\delta}(x_i)\subseteq\bigcup_{i=1}^L B_\epsilon(x_i)$ as $\delta>1$ by construction.
Now suppose $M>N$. Then there exists $a\in A_M$ such that $d(y,a)<\epsilon/(3\delta)$. Then $d(a,\bigcup_{j=1}^N A_j)\le\sum_{k=N}^M r_k$ as $a\in A_M$ (by assumption on the A_{n+1} being contained in $A_n+\overline{B_{r_k}(0)}$). Furthermore, by choice of N, we have $\sum_{k=N}^M r_k<\sum_{k=1}^\infty r_k-\sum_{k=1}^{N-1} r_k<\frac{\epsilon}{3\delta}$. As such, we have $d(a,x_i)\le\frac{\epsilon}{3\delta}+\frac{\epsilon}{3\delta}$ for some $i\in\{1,\dots,L\}$ as the $B_{\epsilon/(3\delta)}(x_i)$ cover $\bigcup_{j=1}^N A_j$. For this i, we have

$$d(y,x_i)\le d(y,a)+d(a,x_i)<\epsilon/(3\delta)+2\epsilon/(3\delta)=\epsilon/\delta<\epsilon$$

as $\delta>1$, so $y\in B_\epsilon(x_i)$.
Thus, we conclude that A is covered by $B_\epsilon(x_1),\dots,B_\epsilon(x_L)$, and A is totally bounded. As A was already shown to be complete, it is compact by the extension of the Heine-Borel theorem.

(b) Let $p \geq 1$ and let $\{r_k\}$ be a sequence in $\mathbb{R}$ such that $r_k > 0$ for all $k \in \mathbb{N}$ and $\sum_{k=1}^{\infty} r_k < \infty$. We show that

$$K = \{x = \{x_k\} \in \ell^p : |x_k| \leq r_k \text{ for all } k \in \mathbb{N}\}$$

is compact. Define the sets $A_1, A_2, \dots$ by

$$A_k = \{x = \{x_n\} \in \ell^p : |x_n| \leq r_n \text{ for all } 1 \leq n \leq k, \text{ and } |x_n| \leq r_n/(n-1) \text{ for all } n > k\}$$

We show that these sets $A_1, A_2, \dots$ satisfy the assumptions in part (a). We start by showing that each A_k is compact. Clearly, each A_k is closed as it is defined by equalities, so it must be complete (as l^p is complete). We now show that A_k is totally bounded. Let $\epsilon > 0$ be given, and choose $K > k$ with $1/K < \epsilon$, and $\sum_{j=1}^{\infty} r_j - \sum_{j=1}^{K} r_j < \epsilon$. Furthermore, assume without loss of generality that $r_l < 1$ for all $l \geq K$. For each $1 \leq j \leq K$, let Y_j a finite set of $\{y_{j,1}, \dots, y_{j,L_j}\} \subset \mathbb{F}$ such that $\{p \in \mathbb{F} : |p| \leq r_j\} \subseteq \bigcup_{i=1}^{L_j} B_{\min\{1/2,\epsilon\}/K}(y_{j,i})$, and $|y_{j,i}| \leq r_j$ for each i. Such a set necessarily exists, as $\{p \in \mathbb{F} : |p| \leq r_j\}$ is clearly compact in $\mathbb{F}$ (as it is closed and bounded), so we can use the total boundedness property. Then the set $Y = Y_1 \times \cdots \times Y_K \times \{0\} \times \{0\} \cdots \subseteq \ell^p$ is also finite. We show that $A_k \subseteq \bigcup_{y \in Y} B_{2\epsilon^{1/p}}(y)$. Let $x \in A_k$; then set $x' = (x_1, \dots, x_K, 0, 0, \dots) \in \ell^p$. There exists some $y \in Y$ such that $\sum_{n=1}^{K} |x'_n - y_n| < K \min\{1/2, \epsilon\}/K = \min\{1/2, \epsilon\}$ by choice of the Y_j. Then we have $\|x' - y\|_p^p = \sum_{n=1}^{K} |x'_n - y|^p \leq \sum_{n=1}^{K} |x'_n - y| < K\epsilon/K = \epsilon$. Note that $|x'_n - y_n|^p \leq |x'_n - y_n|$ as $|x'_n - y_n| < \min\{1/2, \epsilon\}/K$ by choice of the Y_n.

We have

$$\begin{aligned} \|x - y\|_p \leq \|x - x'\|_p + \|x' - y\|_p &< \left(\sum_{n=K+1}^{\infty} |x_n|^p \right)^{1/p} + \epsilon^{1/p} \\ &< \left(\sum_{n=K+1}^{\infty} \frac{r_n}{n-1} \right)^{1/p} + \epsilon^{1/p} \\ &< 2\epsilon^{1/p} \end{aligned}$$

by choice of K, so $x \in B_{2\epsilon^{1/p}}(y)$. As such, we can cover A_k by $\bigcup_{y \in Y} B_{2\epsilon^{1/p}}(y)$ and Y is finite, so A_k is totally bounded and compact.

Now, we clearly have $\bigcup_{k=1}^{\infty} A_k = K$, and K is evidently closed, so by part (a), K is compact.