MA3G7 Functional Analysis I
Course Notes (2014-15)

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1 Revision of Metric Spaces

1.1 Notation and Basic Facts

We shall use $\mathbb{R}$ to denote the set of real numbers and $\mathbb{C}$ to denote the set of complex numbers.

The modulus or absolute value on $\mathbb{R}$ is denoted by $|x|$:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

and the distance between two points $x, y \in \mathbb{R}$ is given by $|x - y|$.

The modulus on $\mathbb{C}$ is denoted by $|z|$. If $z = x + iy$ (with $x, y \in \mathbb{R}$) then

$$|z| = (x^2 + y^2)^{1/2}.$$

(Note that, if $z \in \mathbb{R}$, this agrees with the definition above.) Again, we may define the distance between $z, w \in \mathbb{C}$ by $|z - w|$.

For $n \geq 1$, we shall let $\mathbb{R}^n$ denote the n-dimensional real space and $\mathbb{C}^n$ the n-dimensional complex space:

$$\mathbb{R}^n = \{(x_1, \ldots, x_n) : x_1, \ldots, x_n \in \mathbb{R}\}$$

and

$$\mathbb{C}^n = \{(z_1, \ldots, z_n) : z_1, \ldots, z_n \in \mathbb{C}\}.$$

Write $x = (x_1, \ldots, x_n)$. The natural norm on $\mathbb{R}^n$ (or $\mathbb{C}^n$) is defined by

$$\|x\| = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2}.$$

We shall sometimes denote this norm by $\| \cdot \|_2$ when we want to distinguish it from other norms; this will be made clear at the time.

We may use the norm to define a distance function (or metric) $d(\cdot, \cdot)$ on $\mathbb{R}^n$ or $\mathbb{C}^n$:

$$d(x, y) = \|x - y\| = \left(\sum_{i=1}^{n} |x_i - y_i|^2 \right)^{1/2}.$$

Definition 1.1. A metric space is a set X together with a distance function (or metric) $d : X \times X \to \mathbb{R}$ such that

(1) $d(x, x) = 0$ and $d(x, y) > 0$ if $x \neq y$;

(2) $d(x, y) = d(y, x)$;

(3) $d(x, y) \leq d(x, z) + d(z, y)$.

For $x \in X$ and $\epsilon > 0$, we define the ϵ-ball $B(x, \epsilon)$ centred at x by

$$B(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}.$$

Here are some basic definitions and properties of metric spaces.

- We say that $E \subset X$ is *dense* in X if, for every $x \in X$ and $\epsilon > 0$, there exists $y \in E$ such that $y \in B(x, \epsilon)$.
- We call a set $U \subset X$ an *open* set if, for every $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subset U$. (Note that the empty set $\varnothing$ and X itself are open sets.)
- We call a set $C \subset X$ a *closed* set if C contains its limit points, i.e., if $\{x_n\}_{n=1}^{\infty} \subset C$ and $\lim_{n\to+\infty} x_n = x$ then $x \in C$.
- The *closure* of a set $E \subset X$, denoted by $\overline{E}$, is the union of E and all its limit points (or, equivalently, it is the smallest closed set containing E). The set E is dense in X if and only if $\overline{E} = X$.
- We say that a sequence $\{x_n\}_{n=1}^{\infty}$ in X is a *Cauchy sequence* if, for all $\epsilon > 0$, there exists $N \geq 1$ such that if $n, m \geq N$ then $d(x_n, x_m) < \epsilon$.
- We say that X is *complete* if every Cauchy sequence $\{x_n\}_{n=1}^{\infty}$ in X converges, i.e., there exists $x \in X$ such that $\lim_{n\to+\infty} d(x_n, x) = 0$. More generally, we say that $E \subset X$ is complete if every Cauchy sequence in E converges to a point in E.
- If X is complete and $C \subset X$ is closed then C is complete. (If $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in C then, since X is complete, it converges to some $x \in X$. However, since C is closed, $x \in C$.)
- Let K be a subset of X. If $\{U_i\}_{i\in\mathcal{I}}$ is a collection of open sets such that $K \subset \bigcup_{i\in\mathcal{I}} U_i$ then we call $\{U_i\}_{i\in\mathcal{I}}$ an open cover of K. (Here $\mathcal{I}$ is a possibly infinite index set.) We say that K is a *compact* set if any open cover of K, as above, has a finite subcover, i.e., there exists $\{U_1, \ldots, U_n\} \subset \{U_i\}_{i\in\mathcal{I}}$ such that $K \subset \bigcup_{j=1}^{n} U_j$.
- We say that K is *sequentially compact* if every sequence in K has a subsequence which converges in K. If K is a subset of a metric space then K is sequentially compact if and only if K is compact.
- If K is compact then K is complete. (We do not need to assume that X is complete.)

- Suppose that Y is another metric space. A function $f : X \to Y$ is continuous if, for every open set $U \subset Y$, $f^{-1}U \subset X$ is open. This is equivalent to the ϵ-δ definition: $f : X \to Y$ is continuous if, for every $x \in X$ and for every $\epsilon > 0$, there exists $\delta > 0$ such that, for $y \in X$,

$$d_X(x, y) < \delta \quad \implies \quad d_Y(f(x), f(y)) < \epsilon.$$

  We shall denote the set of continuous functions from X to Y by $C(X, Y)$.

- A function $f : X \to Y$ is uniformly continuous if, for every $\epsilon > 0$, there exists $\delta > 0$ such that, for $x, y \in X$,

$$d_X(x, y) < \delta \quad \implies \quad d_Y(f(x), f(y)) < \epsilon.$$

  If X is compact then any continuous function is uniformly continuous.

- If X is compact then any continuous function $f : X \to \mathbb{R}$ (or $\mathbb{C}$) is bounded, i.e., there exists $M \geq 0$ such that $|f(x)| \leq M$, for all $x \in X$.

A more general setting in which we can think about these concepts is that of *topological spaces*. Let X be a set. A topology on X is a collection of sets $\mathcal{U}$ such that

(i) $\varnothing, V \in \mathcal{U}$;

(ii) (closure under finite intersections)

$$U_1, \dots, U_k \in \mathcal{U} \quad \implies \quad \bigcap_{i=1}^{k} U_i \in \mathcal{U};$$

(iii) (closure under arbitrary unions: the index set $\mathcal{A}$ does not even have to be countable)

$$\{U_\alpha\}_{\alpha \in \mathcal{A}} \subset \mathcal{U} \quad \implies \quad \bigcup_{\alpha \in \mathcal{A}} U_\alpha \in \mathcal{U}.$$

We call the pair $(X, \mathcal{U})$ a topological space and the sets in $\mathcal{U}$ open sets.

We can formulate convergence and continuity in the setting of topological spaces.

- A sequence $x_n \in X$ converges to $x \in X$ if, for every $U \in \mathcal{U}$ such that $x \in U$, there exists $N \geq 1$ such that $x_n \in U$ for all $n \geq N$.

- A function $f : X \to Y$ between two topological spaces $(X, \mathcal{U}_X)$ and $(Y, \mathcal{U}_Y)$ is continuous if, for every $V \in \mathcal{U}_Y$, $f^{-1}(V) \in \mathcal{U}_X$.

One should note that in topological spaces sequential compactness is not equivalent to compactness.

If (X, d) is a metric space then the open sets form a topology. Convergence and continuity only depend on the topology (and, indeed, determine it), so, if two metrics determine the same topology (in which case they are said to be equivalent) they have the same convergent sequences and the same continuous functions.

1.2 The Axiom of Choice and Zorn's Lemma

On occasion, we will need to refer to the Axiom of Choice (or its equivalent formulation, Zorn's Lemma). This is independent of the other axioms of set theory and needs to be assumed for some of the results we introduce.

The Axiom of Choice Let $\mathcal{C}$ be any collection of non-empty sets. Then we can choose a set consisting of exactly one element from each of these sets. (More precisely, there is a function $f : \mathcal{C} \to \bigcup_{A\in\mathcal{C}} A$, such that $f(A) \in A$, for all $A \in \mathcal{C}$.)

To state Zorn's Lemma, we need to introduce the concept of a partially ordered set.

Definition 1.2. A set S with a relation $\succ$ is called *partially ordered* (and $\succ$ is called a *partial ordering*) if

(1) $x \succ x$;

(2) $x \succ y$ and $y \succ z \implies x \succ z$;

(3) $x \succ y$ and $y \succ x \implies x = y$.

If either $x \succ y$ or $y \succ x$ then we say that x and y are *comparable*. (In general, two elements need not be comparable.) A subset $T \subset S$ is called a *chain* if every pair of elements in T are comparable. Let $U \subset S$. We say that $x \in S$ is an *upper bound* for U if $x \succ u$, for all $u \in U$. We say that $x \in S$ is a *maximal element* (for S) if $y \in S$, $y \succ x \implies y = x$.

Example 1.3. Let $S = \mathbb{R}$ and let $\succ$ be $\geq$ (the usual inequality). In this example every pair of elements is comparable.

Example 1.4. (This is a less trivial example.) Let X be a set and let $S = \{A : A \subset X\}$ be the set of all subsets of X. For $A, B \in S$, define $A \succ B \iff A \supset B$. It is easy to find pairs of sets which are not comparable (provided X has at least two elements).

The following is equivalent to the Axiom of Choice.

Zorn's Lemma Let S be a partially ordered set in which every chain has an upper bound. Then S has a maximal element.

1.3 Product Spaces and Tychonoff's Theorem

Let $\{(X_\alpha, \mathcal{U}_\alpha)\}_{\alpha\in\mathcal{A}}$ be an arbitrary family of topological spaces. (In particular, the indexing set $\mathcal{A}$ need not be countable.) Consider the cartesian product space

$$X = \prod_{\alpha\in\mathcal{A}} X_\alpha.$$

This is a topological space with respect to the *product topology*: a set $U \subset X$ is open in this topology if, for all $\alpha \in \mathcal{A}$, the set $p_\alpha^{-1}(U)$ is open in X_α, where $p_\alpha : X \to X_\alpha$ is the projection $p_\alpha((x_{\alpha'})_{\alpha'\in\mathcal{A}}) = x_\alpha$. In other words, the product topology is the smallest topology which makes all the maps $p_\alpha : X \to X_\alpha$ continuous.

Later on, we will use the following theorem.

Theorem 1.5 (Tychonoff's Theorem)**.** *If each topological space* $(X_\alpha, \mathcal{U}_\alpha)$ *is compact then* $X = \prod_{\alpha \in \mathcal{A}} X_\alpha$ *is compact with respect to the product topology.*

Proof. Omitted. (See G. F. Simmons, *Introduction to Topology and Modern Analysis*, McGraw-Hill.) □

2 Vector Spaces and Dimension

2.1 Dimension

Let us start with the definition of a vector space.

Definition 2.1. A *vector space* (or *linear space*) over $\mathbb{R}$ (or $\mathbb{C}$) is a non-empty set V and binary operations $V \times V \to V : (x, y) \mapsto x + y$ (addition) and $\mathbb{R} \times V \to V : (\lambda, x) \mapsto \lambda x$ (or $\mathbb{C} \times V \to V : (\lambda, x) \mapsto \lambda x$) (scalar multiplication) such that

(1) $(x + y) + z = x + (y + z)$, for all $x, y, z \in V$;

(2) $x + y = y + x$, for all $x, y \in V$;

(3) there exists $0 \in V$ such that $x + 0 = x$, for all $x \in V$;

(4) for any $x \in V$, there exists $-x \in V$ such that $x + (-x) = 0$;

(5) $\lambda(x + y) = \lambda x + \lambda y$, for all $\lambda \in \mathbb{R}$ (or $\mathbb{C}$), $x, y \in V$;

(6) $(\lambda + \mu)x = \lambda x + \mu x$, for all $\lambda, \mu \in \mathbb{R}$ (or $\mathbb{C}$), $x \in V$;

(7) $\lambda(\mu x) = (\lambda\mu)x$, for all $\lambda, \mu \in \mathbb{R}$ (or $\mathbb{C}$), $x \in V$; and

(8) $1x = x$, for all $x \in V$.

For future use, we make the following definition.

Definition 2.2. Let V and V' be two vector spaces (over the same field). We say that a map $T : V \to V'$ is a linear map if, for all $\lambda, \mu \in \mathbb{R}$ (or $\mathbb{C}$) and all $x, y \in V$, we have

$$T(\lambda x + \mu y) = \lambda T(x) + \mu T(y).$$

A linear map is a vector space homomorphism. If T is a bijection then it is an isomorphism and we say that V and V' are isomorphic. Two isomorphic vector spaces are indistinguishable as vector spaces.

Definition 2.3. We say that a set $\{x_\alpha\}_{\alpha \in I} \subset V$ is *linearly independent* if whenever $\{\alpha_1, \dots, \alpha_k\}$ is a subset of the indices I such that

$$\sum_{i=1}^{k} \lambda_{\alpha_i} x_{\alpha_i} = 0,$$

where the λ_{α_i}'s are scalars, then

$$\lambda_{\alpha_1} = \cdots = \lambda_{\alpha_k} = 0.$$

(Note that only finite linear combinations are used here.) If $\{x_\alpha\}_{\alpha \in I}$ is not linearly independent then we say that it is linearly dependent.

Definition 2.4. We say that a set $\{x_\alpha\}_{\alpha\in I} \subset V$ *spans* V if every $x \in V$ can be written in the form

$$x = \sum_{i=1}^{k} \lambda_{\alpha_i} x_{\alpha_i},$$

for some indices $\{\alpha_1, \dots, \alpha_k\}$ and scalars λ_{α_i}. (Note that only finite linear combinations are used here.)

Definition 2.5. If $\{x_\alpha\}_{\alpha\in I} \subset V$ is linearly independent and spans V then we say that it is a *Hamel basis* for V.

We say that V is *finite dimensional* if it has a finite Hamel basis $\{x_1, \dots, x_n\}$, say. In finite dimensions, a Hamel basis is just a standard vector space basis. Part (i) in the next lemma shows that any other basis also has n elements and we say that that V has dimension n ($\dim V = n$). If V is not finite dimensional then we say that V is *infinite dimensional*.

Lemma 2.6. *Let V be a vector space.*
(i) Suppose that $\{x_1, \dots, x_n\}$ is a basis for V. If $\{y_1, \dots, y_m\}$ is another basis for V then $n = m$.
(ii) Suppose that V has dimension n. Then any set in V containing at least $n + 1$ elements is linearly dependent.

Proof. These are basic results in linear algebra. □

Example 2.7. $\mathbb{R}^n$ and $\mathbb{C}^n$ are finite dimensional and have dimension n. The standard basis is given by $\{e_i\}_{i=1}^n$, where $e_i = (0, \dots, 0, 1, 0, \dots, 0)$ (with the 1 in the ith place). (Here, of course, we are considering $\mathbb{C}^n$ as a vector space over $\mathbb{C}$. $\mathbb{C}^n$ is also a vector space over $\mathbb{R}$, in which case the dimension is $2n$.)

However, from the point of view of analysis, the most interesting spaces are infinite dimensional.

Example 2.8. Let $V = C([a, b], \mathbb{R})$, where $a < b$. Addition and scalar multiplication are defined pointwise: for $f, g \in C([0, 1], \mathbb{R})$ and $\lambda \in \mathbb{R}$,

$$(f + g)(x) = f(x) + g(x),$$

$$(\lambda f)(x) = \lambda f(x).$$

We claim that $C([a, b], \mathbb{R})$ is infinite dimensional. To see this, we shall find an infinite subset of $C([a, b], \mathbb{R})$ which is linearly independent. Consider $\{p_n\}_{n=0}^\infty$, where $p_n(x) = x^n$. If, for all $x \in [a, b]$,

$$\sum_{i=1}^{k} \lambda_{n_i} x^{n_i} = 0$$

then $\lambda_{n_1} = \cdots = \lambda_{n_k} = 0$, so $\{p_n\}_{n=0}^\infty$ is linearly independent. Thus, $C([a, b], \mathbb{R})$ cannot be finite dimensional: if it had dimension n then any set containing more that n elements would be linearly dependent (Lemma 2.6).

Example 2.9. $R[a,b]$ = the space of Riemann integrable functions $f : [a,b] \to \mathbb{R}$. This is an infinite dimensional space since it contains $C([a,b],\mathbb{R})$.

Example 2.10. Let

$$\ell^1 = \left\{ (x_i)_{i=0}^{\infty} : \sum_{i=0}^{\infty} |x_i| < +\infty,\ x_i \in \mathbb{C} \right\}.$$

(Check that ℓ^1 is a vector space.) Then ℓ^1 is infinite dimensional. To see this consider $\{e_n\}_{n=0}^{\infty}$, where $e_n = (0, \ldots, 0, 1, 0, \ldots)$ (with the 1 in the nth place). Then

$$\sum_{i=1}^{k} \lambda_{n_i} e_{n_i}$$

is the sequence with λ_{n_i} in the n_ith place, $i = 1, \ldots, k$, and zeros elsewhere, so if

$$\sum_{i=1}^{k} \lambda_{n_i} e_{n_i} = 0$$

then we must have $\lambda_{n_1} = \cdots = \lambda_{n_k} = 0$. Therefore $\{e_n\}_{n=0}^{\infty}$ is an infinite linearly independent set in ℓ^1, so ℓ^1 is infinite dimensional.

Remark 2.11. If instead we want to consider real sequences above, we call the resulting space $\ell^1(\mathbb{R})$ and do the same for the analogous spaces below. If there is any risk of confusion, we write $\ell^1(\mathbb{C}) = \ell^1$.

Example 2.12. Let

$$\ell^{\infty} = \left\{ (x_i)_{i=0}^{\infty} : \sup_{0 \le i < \infty} |x_i| < +\infty,\ x_i \in \mathbb{C} \right\}.$$

(Check that ℓ^{∞} is a vector space). Then ℓ^{∞} is infinite dimensional: as for ℓ^1 the set $\{e_n\}_{n=0}^{\infty}$ provides an infinite linearly independent set.

Remark 2.13. We should remark that every vector space has a Hamel basis. (The proof, which we omit, depends on Zorn's Lemma.) However, this is not a very useful result and we included the discussion of Hamel bases so as to be able to define what it means for spaces to be finite or infinite dimensional. When we discuss Hilbert spaces later, there will be a more useful notion of basis.

2.2 Norms on Vector Spaces

Definition 2.14. A norm on a vector space V over $\mathbb{R}$ (or $\mathbb{C}$) is a function $\|\cdot\| : V \to \mathbb{R}$ such that

(1) $\|x\| \ge 0$ and $\|x\| = 0$ if and only if $x = 0$;

(2) $\|\lambda x\| = |\lambda| \|x\|$, for all $x \in V$ and $\lambda \in \mathbb{R}$ (or $\mathbb{C}$);

(3) $\|x+y\| \leq \|x\| + \|y\|$, for all $x, y \in V$.

Example 2.15. Let us first consider norms on $\mathbb{R}^n$ or $\mathbb{C}^n$. The most obvious norm is the Euclidean norm or 2-norm:

$$\left\|\sum_{i=1}^{n} \lambda_i e_i\right\|_2 = \left(\sum_{i=1}^{n} |\lambda_i|^2\right)^{1/2}.$$

Parts (1) and (2) of the definition are obvious. The third part follows from the Cauchy-Schwarz inequality below.

Lemma 2.16 (Cauchy-Schwarz Inequality)**.** *Suppose* $a_i, b_i \in \mathbb{R}$, $i = 1, \ldots, k$. *Then*

$$\sum_{i=1}^{k} a_i b_i \leq \left(\sum_{i=1}^{k} a_i^2\right)^{1/2} \left(\sum_{i=1}^{k} b_i^2\right)^{1/2}.$$

Proof. We have

$$\sum_{i=1}^{k} (a_i t + b_i)^2 \geq 0$$

for all $t \in \mathbb{R}$. By multiplying out each bracket, we may rewrite this inequality as

$$At^2 + 2Bt + C \geq 0,$$

where

$$A = \sum_{i=1}^{k} a_i^2, \quad B = \sum_{i=1}^{k} a_i b_i, \quad C = \sum_{i=1}^{k} b_i^2.$$

If $A > 0$ then the desired inequality follows by taking $t = -B/A$, so that $B^2 - AC \leq 0$. If $A = 0$ then, by varying t, it is easy to see that $B = 0$, as well. □

We can use the Cauchy-Schwarz inequality to check that $\|\cdot\|$ satisfies the triangle inequality on $\mathbb{R}^n$:

$$\begin{aligned}
\left\|\sum_{i=1}^{n} (\lambda_i + \mu_i) e_i\right\|_2^2 &= \sum_{i=1}^{n} |\lambda_i + \mu_i|^2 = \sum_{i=1}^{n} \{|\lambda_i|^2 + 2\lambda_i \mu_i + |\mu_i|^2\} \\
&\leq \sum_{i=1}^{n} |\lambda_i|^2 + 2\left(\sum_{i=1}^{n} \lambda_i^2\right)^{1/2} \left(\sum_{i=1}^{n} \mu_i^2\right)^{1/2} + \sum_{i=1}^{n} |\mu_i|^2 \\
&= \left\{\left(\sum_{i=1}^{n} \lambda_i^2\right)^{1/2} + \left(\sum_{i=1}^{n} \mu_i^2\right)^{1/2}\right\}^2 \\
&= \left\{\left\|\sum_{i=1}^{n} \lambda_i e_i\right\|_2 + \left\|\sum_{i=1}^{n} \mu_i e_i\right\|_2\right\}^2.
\end{aligned}$$

There are other norms which can be defined on $\mathbb{R}^n$. In each case, parts (1) and (2) of the definition of norm is clearly satisfied.

(1) the "1-norm"

$$\left\|\sum_{i=1}^n \lambda_i e_i\right\|_1 = \sum_{i=1}^n |\lambda_i|.$$

For part (3) we just use the usual triangle inequality:

$$\left\|\sum_{i=1}^n (\lambda_i + \mu_i) e_i\right\|_1 = \sum_{i=1}^n |\lambda_i + \mu_i| \le \sum_{i=1}^n (|\lambda_i| + |\mu_i|) = \left\|\sum_{i=1}^n \lambda_i e_i\right\|_1 + \left\|\sum_{i=1}^n \mu_i e_i\right\|_1.$$

(2) "supremum norm" or "∞-norm"

$$\left\|\sum_{i=1}^n \lambda_i e_i\right\|_\infty = \max_{1\le i\le n} |\lambda_i|.$$

Part (3) of the definition is easily verified.

(3) the "p-norm"

$$\|(x_1, \dots, x_n)\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}.$$

Of course, $p = 2$ and $p = 1$ are above. For general p, the triangle inequality follows from Minkowski's inequality below. Note that one has

$$\|(x_1, \dots, x_n)\|_\infty = \lim_{p\to\infty} \|(x_1, \dots, x_n)\|_p,$$

which motivates the notation $\|\cdot\|_\infty$.

Lemma 2.17 (Minkowski's inequality)**.** *For $p \ge 1$,*

$$\left(\sum_{i=1}^n |a_i + b_i|^p\right)^{1/p} \le \left(\sum_{i=1}^n |a_i|^p\right)^{1/p} + \left(\sum_{i=1}^n |b_i|^p\right)^{1/p}.$$

To prove Minkowski's inequality, we need the following generalization of the Cauchy-Schwarz inequality.

Lemma 2.18 (Hölder's Inequality)**.** *Suppose that $p, q > 1$ satisfy $1/p + 1/q = 1$. Then, for $a_i, b_i \in \mathbb{C}$, $i = 1, \dots, n$,*

$$\sum_{i=1}^n |a_i b_i| \le \left(\sum_{i=1}^n |a_i|^p\right)^{1/p} \left(\sum_{i=1}^n |b_i|^q\right)^{1/q},$$

with equality if and only if $|a_i|^p/|b_i|^q$ is constant.

Proof of Hölder's inequality. **Step 1.** We shall show that if $a, b > 0$ and $0 < \lambda < 1$ then

$$a^\lambda b^{1-\lambda} \leq \lambda a + (1-\lambda)b$$

with equality if and only if $a = b$. Set $t = a/b$; then (dividing by b) the result is equivalent to showing that $t^\lambda \leq \lambda t + (1-\lambda)$.

Set $\phi(t) = \lambda t + (1-\lambda) - t^\lambda$; then we need to show that $\phi(t) \geq 0$. However, $\phi'(t) = \lambda - \lambda t^{\lambda-1} = \lambda(1 - t^{\lambda-1})$, so

$$\phi'(t) \begin{cases} < 0 & \text{if } t < 1 \\ = 0 & \text{if } t = 1 \\ > 0 & \text{if } t > 1. \end{cases}$$

Since $\phi(1) = 0$, this gives the result.
Step 2. Define

$$A_i = \frac{|a_i|^p}{\sum_{i=1}^n |a_i|^p}, \quad B_i = \frac{|b_i|^q}{\sum_{i=1}^n |b_i|^q}.$$

Let $\lambda = 1/p$; then, by Step 1,

$$A_i^{1/p} B_i^{1/q} \leq \frac{A_i}{p} + \frac{B_i}{q}$$

(with equality if and only if $|a_i|^p/|b_i|^q = \left(\sum_{i=1}^n |a_i|^p\right) / \left(\sum_{i=1}^n |b_i|^q\right)$, a constant). Writing this out fully, we get, for any $i = 1, \ldots, n$,

$$\frac{|a_i|}{\left(\sum_{i=1}^n |a_i|^p\right)^{1/p}} \frac{|b_i|}{\left(\sum_{i=1}^n |b_i|^q\right)^{1/q}} \leq \frac{1}{p} \frac{|a_i|^p}{\sum_{i=1}^n |a_i|^p} + \frac{1}{q} \frac{|b_i|^q}{\sum_{i=1}^n |b_i|^q}.$$

Summing over $i = 1, \ldots, n$, we obtain

$$\frac{\sum_{i=1}^n |a_i||b_i|}{\left(\sum_{i=1}^n |a_i|^p\right)^{1/p} \left(\sum_{i=1}^n |b_i|^q\right)^{1/q}} \leq \frac{1}{p} + \frac{1}{q} = 1$$

(with equality if and only if $|a_i|^p/|b_i|^q$ is constant). This completes the proof of Hölder's inequality. $\square$

Proof of Minkowski's inequality. The case $p = 1$ is clear, so suppose $p > 1$ and define $q > 1$ by $1/p + 1/q = 1$. We have that

$$\begin{aligned}
&\sum_{i=1}^n |a_i + b_i|^p = \sum_{i=1}^n |a_i + b_i||a_i + b_i|^{p-1} \\
&\leq \sum_{i=1}^n |a_i||a_i + b_i|^{p-1} + \sum_{i=1}^n |b_i||a_i + b_i|^{p-1} \\
&\leq \left(\sum_{i=1}^n |a_i|^p\right)^{1/p} \left(\sum_{i=1}^n |a_i + b_i|^{(p-1)q}\right)^{1/q} + \left(\sum_{i=1}^n |b_i|^p\right)^{1/p} \left(\sum_{i=1}^n |a_i + b_i|^{(p-1)q}\right)^{1/q}
\end{aligned}$$

(using Hölder's inequality). However,

$$(p-1)q = (p-1)\left(1-\frac{1}{p}\right)^{-1} = p,$$

so we may rewrite the above inequality as

$$\sum_{i=1}^{n} |a_i + b_i|^p \leq \left(\left(\sum_{i=1}^{n} |a_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |b_i|^p\right)^{1/p}\right)\left(\sum_{i=1}^{n} |a_i + b_i|^p\right)^{1/q}.$$

Dividing by $\left(\sum_{i=1}^{n} |a_i + b_i|^p\right)^{1/q}$, we get

$$\left(\sum_{i=1}^{n} |a_i + b_i|^p\right)^{1-1/q} \leq \left(\sum_{i=1}^{n} |a_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |b_i|^p\right)^{1/p}.$$

Since $1 - 1/q = 1/p$, this is Minkowski's inequality. $\square$

Now we can go back to the p-norm. Using Minkowski's inequality, we have that

$$\begin{aligned}
\left\|\sum_{i=1}^{n} (\lambda_i + \mu_i)e_i\right\|_p &= \left(\sum_{i=1}^{n} |\lambda_i + \mu_i|^p\right)^{1/p} \\
&\leq \left(\sum_{i=1}^{n} |\lambda_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |\mu_i|^p\right)^{1/p} \\
&= \left\|\sum_{i=1}^{n} \lambda_i e_i\right\|_p + \left\|\sum_{i=1}^{n} \mu_i e_i\right\|_p,
\end{aligned}$$

so $\|\cdot\|_p$ is a norm on $\mathbb{R}^n$ or $\mathbb{C}^n$.

There are natural infinite dimensional analogues for all of these – but notice that we have to use the correct space each time ($\ell^2, \ell^1, \ell^\infty, \ldots$) to have the norm defined.

Example 2.19. $\ell^2 = \left\{(x_i)_{i=0}^\infty : \ \sum_{i=0}^\infty |x_i|^2 < +\infty, \ x_i \in \mathbb{C}\right\}$ (which is an infinite dimensional space for the same reason that ℓ^1 is) with the norm

$$\|(x_i)_{i=0}^\infty\|_2 = \left(\sum_{i=0}^{\infty} |x_i|^2\right)^{1/2}.$$

To prove that this satisfies the triangle inequality, notice that the Cauchy-Schwarz inequality continues to hold for infinite sums, with the same proof, provided that everything converges.

Example 2.20. $\ell^1 = \left\{(x_i)_{i=0}^\infty : \ \sum_{i=0}^\infty |x_i| < +\infty, \ x_i \in \mathbb{C}\right\}$ with the norm

$$\|(x_i)_{i=0}^\infty\|_1 = \sum_{i=0}^{\infty} |x_i|.$$

Remark 2.21. We have $\ell^1 \subset \ell^2$ but $\ell^1 \neq \ell^2$.

To see this, first take $(x_i)_{i=1}^\infty \in \ell^1$. Then $\sum_{i=1}^\infty |x_i| < +\infty$ so, in particular, $\lim_{i\to+\infty} x_i = 0$. Thus there exists $N \geq 1$ such that

$$i \geq N \quad \Longrightarrow \quad |x_i| < 1.$$

If $|x_i| < 1$ then $|x_i|^2 < |x_i|$, so this holds for all $i \geq N$. Applying the Comparison Test, we then have that $\sum_{i=1}^\infty |x_i|^2 < +\infty$, so that $(x_i)_{i=1}^\infty \in \ell^2$. Thus $\ell^1 \subset \ell^2$.

On the other hand, consider $(1, \frac{1}{2}, \frac{1}{3}, \ldots)$. We know

$$\sum_{i=1}^\infty \frac{1}{i} = +\infty \quad \text{so} \quad \left(1, \frac{1}{2}, \frac{1}{3}, \ldots\right) \notin \ell^1.$$

However,

$$\sum_{i=1}^\infty \frac{1}{i^2} < +\infty \quad \text{so} \quad \left(1, \frac{1}{2}, \frac{1}{3}, \ldots\right) \in \ell^2.$$

Thus $\ell^1 \neq \ell^2$.

Example 2.22. $\ell^\infty = \left\{(x_i)_{i=0}^\infty : \sup_{0\leq i<\infty} |x_i| < +\infty,\ x_i \in \mathbb{C}\right\}$ with the norm

$$\|(x_i)_{i=0}^\infty\|_\infty = \sup_{0\leq i<\infty} |x_i|.$$

Remark 2.23. ℓ^1, ℓ^∞, ℓ^2 can be viewed as infinite dimensional analogues of $\mathbb{C}^n$, with the norms $\|\cdot\|_1$, $\|\cdot\|_\infty$, $\|\cdot\|_2$, respectively.

Example 2.24. Let $p \geq 1$, $\ell^p = \left\{(x_i)_{i=0}^\infty : \sum_{i=0}^\infty |x_i|^p < +\infty,\ x_i \in \mathbb{C}\right\}$ with the norm

$$\|(x_i)_{i=0}^\infty\|_p = \left(\sum_{i=0}^\infty |x_i|^p\right)^{1/p}.$$

That the triangle inequality holds in this case follows from the fact that Minkowski's inequality continues to hold for infinite sums, provided that everything converges.

Analogous to the result for ℓ^1 and ℓ^2 above, one can show the following. For $1 \leq p < q < +\infty$, $\ell^p \subset \ell^q$ but $\ell^p \neq \ell^q$. Also, $\ell^p \subset \ell^\infty$ but $\ell^p \neq \ell^\infty$. (See exercises.)

Example 2.25. Now we consider the infinite dimensional space $C([a, b], \mathbb{R})$. There are several norms we can define here.

(a) "uniform norm" or "supremum norm" or "∞-norm":

$$\|f\|_\infty = \sup_{x\in[a,b]} |f(x)|.$$

(b) "1-norm":

$$\|f\|_1 = \int_a^b |f(x)|\, dx.$$

(c) "2-norm":

$$\|f\|_2 = \left(\int_a^b |f(x)|^2 \, dx \right)^{1/2}.$$

For part (3), use the Cauchy-Schwarz inequality for integrals:

$$\left| \int_a^b f(x)g(x)dx \right| \le \left(\int_a^b |f(x)|^2 dx \right)^{1/2} \left(\int_a^b |g(x)|^2 dx \right)^{1/2}.$$

(d) "p-norm":

$$\|f\|_p = \left(\int_a^b |f(x)|^p \, dx \right)^{1/p}.$$

For the triangle inequality, use Minkowski's inequality for integrals:

$$\left(\int_a^b |f(x) + g(x)|^p \, dx \right)^{1/p} \le \left(\int_a^b |f(x)|^p \, dx \right)^{1/p} + \left(\int_a^b |g(x)|^p \, dx \right)^{1/p}.$$

We have put $\| \cdot \|_\infty$ first because it is the most important one or, more accurately, it is the one best adapted to the space $C([a, b], \mathbb{R})$.

2.3 Metrics and Topology

Let $\| \cdot \|$ be a norm on the vector space V. This defines a metric on V by $d(x, y) = \|x - y\|$. The axioms for a metric:

(1) $d(x, y) \ge 0$ and $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$; and

(3) $d(x, y) \le d(x, z) + d(z, y)$

follow from the corresponding properties of the norm $\| \cdot \|$.

Suppose we have two norms, $\| \cdot \|$ and $\| \cdot \|'$, on V. They each give rise to a topology on V: can these topologies be the same (i.e. is it possible that a set is open for one norm if and only if it is open for the other)? A necessary and sufficient condition is given by the following.

Definition 2.26. We say that two norms $\| \cdot \|$ and $\| \cdot \|'$ are equivalent if there exist $C_1, C_2 > 0$ such that

$$C_1 \|x\|' \le \|x\| \le C_2 \|x\|',$$

for all $x \in V$.

(This is obviously an equivalence relation. In particular, if $\| \cdot \|, \| \cdot \|'$ are equivalent and $\| \cdot \|', \| \cdot \|''$ are equivalent then $\| \cdot \|, \| \cdot \|''$ are equivalent.)

Lemma 2.27. *Two norms on V give the same topology if and only if they are equivalent.*

Proof. Suppose that $\|\cdot\|$ and $\|\cdot\|'$ are equivalent. Consider the identity map

$$I : (V, \|\cdot\|) \to (V, \|\cdot\|') : x \mapsto x.$$

Given $\epsilon > 0$, take $\delta = C_1\epsilon$. Then $d(x,y) = \|x-y\| < \delta \implies d'(x,y) = \|x-y\|' \leq C_1^{-1}\|x-y\| < C_1^{-1}\delta = \epsilon$, so I is continuous. Hence, if $U \subset V$ is open for $\|\cdot\|'$, then $I^{-1}(U)$ is open for $\|\cdot\|$. But $I^{-1}(U) = U$, so U is open for $\|\cdot\|$.

Similarly, by considering

$$J : (V, \|\cdot\|') \to (V, \|\cdot\|) : x \mapsto x,$$

we can also show that every $\|\cdot\|$-open set is also $\|\cdot\|$-open.

Now suppose that $\|\cdot\|$ and $\|\cdot\|'$ are not equivalent. Then, without loss of generality, there exits a sequence of points $x_n \in V$, $n \geq 1$, such that $\|x_n\|/\|x_n\|' \to 0$, as $n \to +\infty$. Set $y_n = x_n/\|x_n\|'$. Then $y_n \to 0$ in $(V, \|\cdot\|)$ but $I(y_n)$ does not converge to 0 in $(V, \|\cdot\|')$ (since $\|y_n\|' = 1$ for all n). Thus, I is not continuous, so there exist U which is $\|\cdot\|'$-open such that $U = I^{-1}(U)$ is not $\|\cdot\|$-open, i.e., the topologies are different. □

Example 2.28. We can check that $(\mathbb{R}^n, \|\cdot\|_1)$ and $(\mathbb{R}^n, \|\cdot\|_2)$ are equivalent. As we shall see in a moment, this is part of a more general phenomenon.

Write $x = (x_1, \ldots, x_n)$. Then

$$\|x\|_2^2 = |x_1|^2 + \cdots + |x_n|^2 \leq (|x_1| + \cdots + |x_n|)^2 = \|x\|_1^2,$$

so

$$\|x\|_2 \leq \|x\|_1.$$

On the other hand,

$$\begin{aligned}\|x\|_1^2 &= (|x_1| + \cdots + |x_n|)^2\\ &= |x_1|^2 + \cdots |x_n|^2 + 2|x_1x_2| + 2|x_1x_3| + \cdots + 2|x_{n-1}x_n|.\end{aligned}$$

Now, for $i \neq j$,

$$|x_i|^2 - 2|x_ix_j| + |x_j|^2 = (|x_i| - |x_j|)^2 \geq 0,$$

so

$$2|x_ix_j| \leq |x_i|^2 + |x_j|^2 \leq \|x\|_2^2.$$

Thus

$$\|x\|_1^2 \leq \|x\|_2^2 + \frac{n(n-1)}{2}\|x\|_2^2 < n^2\|x\|_2^2,$$

so

$$\|x\|_1 \leq n\|x\|_2.$$

Lemma 2.29. *On $\mathbb{R}^n$ (and $\mathbb{C}^n$), all norms are equivalent.*

Proof. Let $\|\cdot\|_1$ be the 1-norm on $\mathbb{R}^n$ and let $\|\cdot\|$ be an arbitrary norm. We shall show that $\|\cdot\|_1$ and $\|\cdot\|$ are equivalent.

Write $x = \sum_{i=1}^n a_i e_i$ and let $M = \max_{1\le i\le n} \|e_i\|$. Then

$$\|x\| \le \sum_{i=1}^n |a_i|\|e_i\| \le M\sum_{i=1}^n |a_i| = M\|x\|_1.$$

Now we shall show that $\inf_{x\in\mathbb{R}^n, x\neq 0} \|x\|/\|x\|_1$ is positive. If it isn't, then we can find a sequence x_i such that $\lim_{i\to+\infty} \|x_i\|/\|x_i\|_1 = 0$. Set $y_i = x_i/\|x_i\|_1$, so $y_i \in \{y \in \mathbb{R}^n : \|y\|_1 \le 1\}$. This set is closed and bounded, so (by the Bolzano-Weierstrass Theorem) y_i has a convergent subsequence y_{i_j} with limit y. In other words $\lim_{j\to+\infty} \|y_{i_j} - y\|_1 = 0$ and (since $\|y_{i_j}\|_1 = 1$) $\|y\|_1 \neq 0$. However, we also have

$$|\|y_{i_j}\| - \|y\|| \le \|y_{i_j} - y\| \le M\|y_{i_j} - y\|_1 \to 0, \text{ as } j \to +\infty,$$

so $\|y\| = \lim_{j\to+\infty} \|y_{i_j}\| = 0$. But $\|y\| = 0$ if and only if $y = 0$, giving a contradiction to $\|y\|_1 \neq 0$. Therefore, we can define

$$0 < m = \inf_{x\in\mathbb{R}^n} \frac{\|x\|}{\|x\|_1}.$$

Clearly, $\|x\| \ge m\|x\|_1$, as required. □

However, this result is not true for infinite dimensional spaces.

Example 2.30. Consider the space $C([0,1],\mathbb{R})$. Then the two norms $\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|$ and $\|f\|_1 = \int_0^1 |f(x)|dx$ are *not* equivalent. For example, consider the sequence

$$f_n(x) = \begin{cases} 1 - nx & \text{if } 0 \le x \le 1/n \\ 0 & \text{if } 1/n < x \le 1. \end{cases}$$

Then $\|f_n\|_\infty = 1$, for all n, but

$$\|f\|_1 = \int_0^{1/n} (1 - nx)dx = \left[x - \frac{nx^2}{2}\right]_0^{1/n} = \frac{1}{n} - \frac{1}{2n} = \frac{1}{2n} \to 0,$$

as $n \to +\infty$.

2.4 The unit ball

Let $(V, \|\cdot\|)$ be a normed vector space and let B denote the closed unit ball, i.e.

$$B = \{x \in V : \|x\| \le 1\}.$$

If V is finite dimensional then, by the Heine-Borel Theorem, B is compact. However, this is the only situation in which B is compact.

Theorem 2.31. *The closed unit ball in a normed vector space $(V, \|\cdot\|)$ is compact if and only if V is finite dimensional.*

To prove the theorem we need the following lemma.

Lemma 2.32. *Let $(V, \|\cdot\|)$ be a normed vector space, $S \subsetneq V$ a (proper) closed linear subspace of V. Then there exists $\bar{x} \in V$ such that $\|\bar{x}\| = 1$ and $\|\bar{x} - y\| > 1/2$ for any $y \in S$.*

Proof. This proof is constructive and we can find the point $\bar{x}$ explicitly. Because $S \subsetneq V$, there exists at least one $\tilde{x} \in V \setminus S$. Since S is closed,

$$M := \inf_{y \in S} \{\|\tilde{x} - y\|\} > 0\,.$$

Now take $y_0 \in S$ such that $\|\tilde{x} - y_0\| < 2M$. Then the required $\bar{x}$ is simply $\bar{x} = \frac{\tilde{x} - y_0}{\|\tilde{x} - y_0\|}$. Clearly, $\bar{x} = 1$. Also, for any $y \in S$ we have

$$\begin{aligned}
\|y - \bar{x}\| &= \left\| y - \frac{\tilde{x} - y_0}{\|\tilde{x} - y_0\|} \right\| = \frac{\|y\|\tilde{x} - y_0\| + y_0 - \tilde{x}\|}{\|\tilde{x} - y_0\|} \\
&= \frac{\|y' - \tilde{x}\|}{\|\tilde{x} - y_0\|} > \frac{M}{\|\tilde{x} - y_0\|} > \frac{M}{2M} = \frac{1}{2},
\end{aligned}$$

having set $y' = y\|\tilde{x} - y_0\| + y_0 \in S$. $\square$

Proof of Theorem 2.31. Suppose that V is infinite dimensional. Let $\{x_n\} \subset V$ be a countable collection of elements of V such that any finite subset of $\{x_n\}$ consists of linearly independent elements. For any $m \geq 1$ let $E_m := \operatorname{span}\{x_1 \dots x_m\}$. Then $E_m \subsetneq E_{m+1}$, E_m is a linear subspace of E_{m+1} and it is closed (because it is isomorphic to $\mathbb{R}^m$). Then we can use Lemma 2.32 and construct an element $y_{m+1} \in (E_{m+1} \backslash E_m)$ such that $\|y_{m+1}\| = 1$ and $\|y_{m+1} - y\| > 1/2$ for any $y \in E_m$. Hence, by construction, $\|y_{k+1} - y\| > 1/2$ for any $y \in E_k$, $k \leq m$. In this way we have constructed a sequence $\{y_m\}$ of elements of the closed unit ball such that $\|y_m - y_k\| > 1/2$ for any $m \neq k$. The sequence $\{y_m\}$ has no convergent subsequence hence the closed unit ball is not sequentially compact. $\square$

3 Continuous functions.

3.1 Weierstrass Approximation Theorem

Let us consider again the vector space $C([a,b],\mathbb{R})$ of continuous functions $f:[a,b]\to\mathbb{R}$. A norm on this space is given by

$$\|f\|_\infty = \sup_{x\in[a,b]} |f(x)|;$$

this is called the uniform norm or the supremum norm. (Since $[a,b]$ is compact, f is bounded and so this quantity is finite.) This allows us to define a metric on $C([a,b],\mathbb{R})$ by

$$d_\infty(f,g) = \|f-g\|_\infty.$$

(It is conventional just to use norm notation in this context, so in fact we will write $\|f-g\|_\infty$ and not $d_\infty(f,g)$.)

For convenience, we will restrict to the special case $[a,b]=[0,1]$, although the results below extend to arbitrary intervals. We are going to show that for any such f and any $\epsilon>0$, we can find a polynomial $p(x)=a_0+a_1x+\ldots+a_nx^n$ such that, for all $x\in[0,1]$, $|f(x)-p(x)|\le\epsilon$. (The degree n of the polynomial is not fixed; we can take it as large as we need to get the approximation.)

In terms of the uniform norm $\|\cdot\|_\infty$ introduced above, the approximation condition may be written as $\|f-p\|_\infty\le\epsilon$.

Another way of phrasing the result is that polynomial functions are *uniformly dense* in $C([0,1],\mathbb{R})$, i.e., dense with respect to the metric defined by the uniform norm $\|\cdot\|_\infty$.

Let us state the result formally as a theorem.

Theorem 3.1 (Weierstrass Approximation Theorem)**.** *Suppose that $f\in C([0,1],\mathbb{R})$ and that $\epsilon>0$. Then there exists a polynomial $p(x)$ such that $\|f-p\|_\infty\le\epsilon$.*

In fact, we shall show that f may be approximated by polynomials of a particular form:

$$B_n(f;x) := \sum_{k=0}^{n}\binom{n}{k} f\left(\frac{k}{n}\right) x^k(1-x)^{n-k}.$$

This is called the n^{th} *Bernstein polynomial for f*. Clearly it is a polynomial of degree n.

Before we prove the theorem, we need a lemma.

Lemma 3.2. *(i)*

$$\sum_{k=0}^{n}\binom{n}{k} x^k(1-x)^{n-k} = 1.$$

(ii)

$$\sum_{k=0}^{n} k\binom{n}{k} x^k(1-x)^{n-k} = nx.$$

(iii)

$$\sum_{k=0}^{n}(k-nx)^2\binom{n}{k}x^k(1-x)^{n-k}=nx(1-x).$$

Proof. (i) This follows from the Binomial Theorem.

(ii) Note that

$$\begin{aligned}\frac{d}{dx}\left(x^k(1-x)^{n-k}\right)&=kx^{k-1}(1-x)^{n-k}-(n-k)x^k(1-x)^{n-k-1}\\&=x^k(1-x)^{n-k}\frac{k-nx}{x(1-x)}.\end{aligned}$$

Thus, differentiating (i) gives

$$\begin{aligned}0&=\frac{d}{dx}\left(\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}\right)\\&=\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}\frac{k-nx}{x(1-x)}\end{aligned}$$

and, taking out the factor $1/x(1-x)$,

$$\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}(k-nx)=0. \tag{$*$}$$

Rearranging and using (i),

$$\sum_{k=0}^{n}k\binom{n}{k}x^k(1-x)^{n-k}=nx\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}=nx,$$

as required.

(iii) Now differentiate (ii) to get

$$\begin{aligned}n&=\frac{d}{dx}\left(\sum_{k=0}^{n}k\binom{n}{k}x^k(1-x)^{n-k}\right)\\&=\sum_{k=0}^{n}k\binom{n}{k}x^k(1-x)^{n-k}\frac{k-nx}{x(1-x)}.\end{aligned}$$

Thus

$$\sum_{k=0}^{n}k\binom{n}{k}x^k(1-x)^{n-k}(k-nx)=nx(1-x). \tag{$**$}$$

Note that

$$(k-nx)^2=k(k-nx)-nx(k-nx).$$

Hence, using $(*)$ and $(**)$,

$$\begin{aligned}
&\sum_{k=0}^{n}(k-nx)^2\binom{n}{k}x^k(1-x)^{n-k}\\
&=\sum_{k=0}^{n}k(k-nx)\binom{n}{k}x^k(1-x)^{n-k}-nx\sum_{k=0}^{n}(k-nx)\binom{n}{k}x^k(1-x)^{n-k}\\
&=nx(1-x)+0=nx(1-x),
\end{aligned}$$

as required. □

Proof the Weierstrass Approximation Theorem. We shall now prove the Weierstrass Approximation Theorem. Fix $\epsilon>0$. By uniform continuity of f, there exists $\delta>0$ such that, for $x,y\in[0,1]$,

$$|x-y|<\delta \quad\Longrightarrow\quad |f(x)-f(y)|<\frac{\epsilon}{2}.$$

Using Lemma 3.2(i), we have

$$f(x)-B_n(f;x)=\sum_{k=0}^{n}\left(f(x)-f\left(\frac{k}{n}\right)\right)\binom{n}{k}x^k(1-x)^{n-k}.$$

Thus

$$|f(x)-B_n(f;x)|\leq\sum_{k=0}^{n}\left|f(x)-f\left(\frac{k}{n}\right)\right|\binom{n}{k}x^k(1-x)^{n-k}=\Sigma_1(x)+\Sigma_2(x),$$

where

$$\Sigma_1(x)=\sum_{\substack{0\leq k\leq n\\ k\,:\,\left|x-\frac{k}{n}\right|<\delta}}\left|f(x)-f\left(\frac{k}{n}\right)\right|\binom{n}{k}x^k(1-x)^{n-k}<\frac{\epsilon}{2}$$

and

$$\begin{aligned}
\Sigma_2(x)&=\sum_{\substack{0\leq k\leq n\\ k\,:\,\left|x-\frac{k}{n}\right|\geq\delta}}\left|f(x)-f\left(\frac{k}{n}\right)\right|\binom{n}{k}x^k(1-x)^{n-k}\\
&\leq 2\|f\|_\infty\sum_{k\,:\,(k-nx)^2\geq n^2\delta^2}\binom{n}{k}x^k(1-x)^{n-k}\\
&\leq 2\|f\|_\infty\frac{1}{n^2\delta^2}\sum_{k=0}^{n}(k-nx)^2\binom{n}{k}x^k(1-x)^{n-k}\\
&=2\|f\|_\infty\frac{nx(1-x)}{n^2\delta^2}\\
&\leq\frac{\|f\|_\infty}{2\delta^2 n},
\end{aligned}$$

where we have used Lemma 3.2 (and the easy inequality $x(1-x) \leq 1/4$).

Combining the estimates on $\Sigma_1(x)$ and $\Sigma_2(x)$, we obtain

$$|f(x) - B_n(f;x)| \leq \frac{\epsilon}{2} + \frac{\|f\|_\infty}{2\delta^2 n}.$$

Now choose N sufficiently large that

$$\frac{\|f\|_\infty}{2\delta^2 N} < \frac{\epsilon}{2}.$$

(One may take $N = [\|f\|_\infty/\epsilon\delta^2] + 1$.) Then

$$|f(x) - B_N(f;x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

so $B_N(f;x)$ is a polynomial satisfying the conclusion of the theorem. □

Remark 3.3. The rate of convergence of $B_n(f;\cdot)$ to f is very slow. In fact, if

$$\|f - B_n(f;\cdot)\|_\infty = o(n^{-1})$$

(i.e. $\lim_{n\to+\infty} n\|f - B_n(f;\cdot)\|_\infty = 0$) then f is linear ($f(x) = ax + b$). In contrast, take $f(x) = e^x$ and $p_n(x) = \sum_{k=0}^n \frac{x^k}{k!}$ (*not* a Bernstein polynomial). Then

$$|f(x) - p_n(x)| = \sum_{k=n+1}^\infty \frac{x^k}{k!} \leq \sum_{k=n+1}^\infty \frac{1}{k!} \leq \frac{1}{n!}.$$

This is incredibly fast *but* this function is real analytic, which is very rare.

3.2 The Stone-Weierstrass Theorem

The Weierstrass Approximation Theorem is a special case of a much more general theorem valid for compact metric spaces. If X is a compact metric space, $C(X,\mathbb{R})$ will denote the set of continuous functions $f : X \to \mathbb{R}$. We can define the uniform norm on $C(X,\mathbb{R})$ by

$$\|f\|_\infty = \sup_{x\in X} |f(x)|.$$

Definition 3.4. We say that $\mathcal{A} \subset C(X,\mathbb{R})$ is an algebra if $\mathcal{A}$ is a linear subspace of $C(X,\mathbb{R})$ with the additional property that

$$f, g \in \mathcal{A} \quad \Longrightarrow \quad fg \in \mathcal{A}.$$

(Here fg is just the function obtained by pointwise multiplication: $(fg)(x) = f(x)g(x)$.)

Theorem 3.5 (Stone-Weierstrass Theorem)**.** *Let X be a compact metric space. Let $\mathcal{A} \subset C(X,\mathbb{R})$ be an algebra such that*

(1) $\mathcal{A}$ contains a non-zero constant function;

(2) $\mathcal{A}$ separates points (i.e., if $x, x' \in X$, $x \neq x'$, then there exists $f \in \mathcal{A}$ such that $f(x) \neq f(x')$.

Then $\mathcal{A}$ is uniformly dense in $C(X, \mathbb{R})$.

Proof. Omitted. □

Remark 3.6. Since $\mathcal{A}$ is an algebra, if $\mathcal{A}$ contains a non-zero constant function then it contains all non-zero constant functions.

Example 3.7. Let $X = [0, 1]$ and take $\mathcal{A}$ to be the set of polynomials on X. Then $\mathcal{A}$ contains the non-zero constant function 1 and the function $p(x) = x \in \mathcal{A}$ separates points, so the hypotheses of Theorem 3.5 are satisfied. This shows that the Weierstrass Approximation Theorem is a special case of the Stone-Weierstrass Theorem. However, the WAT is a component of the proof of the SWT, so the WAT needs to be proved independently.

Example 3.8. Let $X = [0, 1]$ and

$$\mathcal{A} = \left\{ a_0 + \sum_{n=1}^{N} a_n \cos(2\pi n x) + \sum_{n=1}^{M} b_n \sin(2\pi n x) \,:\, a_n, b_n \in \mathbb{R}, N, M \geq 1 \right\}.$$

Then $\mathcal{A}$ satisfies hypothesis (1) (since $1 \in \mathcal{A}$) but not hypothesis (2) (since $f(0) = f(1)$ for all $f \in \mathcal{A}$). In fact, it is easy to see that $\mathcal{A}$ is *not* uniformly dense in $C([0, 1], \mathbb{R})$: choose $g \in C([0, 1], \mathbb{R})$ with $g(0) \neq g(1)$ and suppose that $2\epsilon < |g(0) - g(1)|$. Then, for any $f \in \mathcal{A}$, either $|g(0) - f(0)| \geq \epsilon$ or $|g(1) - f(1)| \geq \epsilon$, so that $\|g - f\|_\infty \geq \epsilon$.

However, the following simple modification of X allows us to make $\mathcal{A}$ a dense set.

Example 3.9. Let $X = [0, 1]$ with 0 and 1 identified. (The parametrization $\{e^{2\pi i x} : 0 \leq x \leq 1\}$ identifies X with the unit circle.) Take $\mathcal{A}$ as in Example 3.8. For $f \in \mathcal{A}$, the fact that $f(0) = f(1)$ now means that f is well-defined as a function on X. Furthermore, one can easily check that $\mathcal{A}$ now satisfies the hypotheses of Theorem 3.5. Therefore $\mathcal{A}$ is uniformly dense in $C(X, \mathbb{R})$. One can identify functions on X with functions on $[0, 1]$ with $f(0) = f(1)$, so the result says that such functions can be uniformly approximated by functions in $\mathcal{A}$. Similarly, continuous functions $f : [-\pi, \pi] \to \mathbb{R}$ with $f(-\pi) = f(\pi)$ can be uniformly approximated by functions in

$$\mathcal{A}' = \left\{ a_0 + \sum_{n=1}^{N} a_n \cos(nx) + \sum_{n=1}^{M} b_n \sin(nx) \,:\, a_n, b_n \in \mathbb{R}, N, M \geq 1 \right\}.$$

This will be important when we discuss Fourier series later.

4 Banach Spaces

4.1 Definition and examples

Recall that a sequence $x_n \in V$ is called a Cauchy sequence if, for every $\epsilon > 0$, there exists $N \geq 1$ such that $n, m \geq N \implies d(x_n, x_m) < \epsilon$. Also recall the following.

Definition 4.1. A metric space V is said to be *complete* if every Cauchy sequence converges in V (i.e. if x_n is a Cauchy sequence in V then there exists $x \in V$ such that $x_n \to x$, as $n \to +\infty$.)

The following is a very important definition.

Definition 4.2. A normed vector space which is also complete (with respect to the metric given by the norm) is called a *Banach space*.

Remark 4.3. Banach spaces are named after Stefan Banach (1892-1945). He introduced them and developed their theory in the 1920s and 1930s.

Example 4.4. $\mathbb{R}^n$ and $\mathbb{C}^n$ (with any norm) are Banach spaces. (This is a standard result in real analysis.)

Theorem 4.5. *$C([a,b],\mathbb{R})$ with the norm $\|f\|_\infty = \sup_{x\in[a,b]} |f(x)|$ is a Banach space. (So is $C([a,b],\mathbb{C})$ with the same proof.)*

Proof. Suppose that $\{f_n\}_{n=1}^\infty$ is a Cauchy sequence for $\|\cdot\|_\infty$. This means that, given $\epsilon > 0$, there exists $N \geq 1$ such that, for $n, m \geq N$ and for any $x \in [a,b]$,

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty < \epsilon. \tag{$*$}$$

In other words, $\{f_n(x)\}_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{R}$ and so, since $\mathbb{R}$ is complete, it has a limit $f(x)$, say. We may now let $m \to +\infty$ in $(*)$ to obtain, for $n \geq N$,

$$|f_n(x) - f(x)| = \lim_{m\to+\infty} |f_n(x) - f_m(x)| \leq \limsup_{m\to+\infty} \|f_n - f_m\|_\infty \leq \epsilon.$$

Now the above inequality tells us that $f_n(x)$ converges to $f(x)$ *uniformly* on $[a,b]$ and so, by a theorem in Real Analysis, f is continuous (i.e., it's an element of $C([a,b],\mathbb{R})$). Furthermore, taking the supremum over $x \in [a,b]$ in the inequality gives that, for $n \geq N$,

$$\|f_n - f\|_\infty = \sup_{x\in[a,b]} |f_n(x) - f(x)| \leq \epsilon,$$

so $\lim_{n\to+\infty} \|f_n - f\|_\infty = 0$, as required. □

Remark 4.6. A similar proof shows that, for any compact metric space X, $C(X,\mathbb{R})$ and $C(X,\mathbb{C})$ are Banach spaces with respect to $\|\cdot\|_\infty$.

Remark 4.7. Note the three steps in the above proof:

(1) identify a potential limit f for the Cauchy sequence f_n;

(2) show that f is in the desired space (here $C([a,b],\mathbb{R})$);

(3) show that f_n converges to f in the appropriate norm (here $\|\cdot\|_\infty$).

We will use this scheme again in the next theorem.

Theorem 4.8. *$\ell^\infty = \left\{(x_i)_{i=0}^\infty : \sup_{0\le i<\infty}|x_i| < +\infty,\ x_i \in \mathbb{C}\right\}$ with the norm $\|(x_i)_{i=0}^\infty\|_\infty = \sup_{0\le i<\infty}|x_i|$ is a Banach space.*

Proof. (This is very similar to the previous example.) Suppose that $\{(x_i^{(n)})_{i=0}^\infty\}_{n=1}^\infty$ is a Cauchy sequence for $\|\cdot\|_\infty$. Then, given $\epsilon > 0$, there exists $N \ge 1$ such that, for $n, m \ge N$ and any $i \ge 0$,

$$|x_i^{(n)} - x_i^{(m)}| \le \|(x_i^{(n)})_{i=0}^\infty - (x_i^{(m)})_{i=1}^\infty\|_\infty < \epsilon. \tag{$*$}$$

In other words, for each fixed i, $\{x_i^{(n)}\}_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{C}$ and so, since $\mathbb{C}$ is complete, it has a limit $x_i \in \mathbb{C}$, say. We may now let $m \to +\infty$ in $(*)$ to obtain, for $n \ge N$,

$$|x_i^{(n)} - x_i| = \lim_{m\to+\infty}|x_i^{(n)} - x_i^{(m)}| \le \limsup_{m\to+\infty}\|(x_i^{(n)})_{i=0}^\infty - (x_i^{(m)})_{i=0}^\infty\|_\infty \le \epsilon. \tag{$**$}$$

Then, in particular, for each i,

$$|x_i| \le |x_i^{(N)}| + |x_i^{(N)} - x_i| \le \|(x_i^{(N)})_{i=1}^\infty\|_\infty + \epsilon,$$

so, taking the supremum over i, $\|(x_i)_{i=0}^\infty\|_\infty < +\infty$, i.e., $(x_i)_{i=0}^\infty \in \ell^\infty$. Furthermore, taking the supremum over i in $(**)$, gives that, for $n \ge N$,

$$\|(x_i^{(n)})_{i=0}^\infty - (x_i)_{i=0}^\infty\|_\infty = \sup_{i\ge 0}|x_i^{(n)} - x_i| \le \epsilon,$$

so $\lim_{n\to+\infty}\|(x_i^{(n)})_{i=0}^\infty - (x_i)_{i=0}^\infty\|_\infty = 0$, as required. □

Theorem 4.9. *ℓ^1 with the norm $\|\cdot\|_1$ is a Banach space.*

Proof. Suppose that $\{(x_i^{(n)})_{i=1}^\infty\}_{n=1}^\infty$ is a Cauchy sequence for $\|\cdot\|_1$. Then, given $\epsilon > 0$, there exists $N \ge 1$ such that, for $n, m \ge N$,

$$\sum_{i=1}^\infty |x_i^{(n)} - x_i^{(m)}| = \|(x_i^{(n)})_{i=1}^\infty - (x_i^{(m)})_{i=1}^\infty\|_1 < \epsilon \tag{$*$}$$

and so, in particular, for any i,

$$|x_i^{(n)} - x_i^{(m)}| < \epsilon.$$

Thus, for each fixed i, $\{x_i^{(n)}\}_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{C}$ and so has a limit $x_i \in \mathbb{C}$.

Next, note that, for any $M \ge 1$ and $n, m \ge N$, we have

$$\sum_{i=1}^M |x_i^{(n)} - x_i^{(m)}| \le \sum_{i=1}^\infty |x_i^{(n)} - x_i^{(m)}| < \epsilon.$$

If we let $m \to +\infty$, this gives

$$\sum_{i=1}^{M} |x_i^{(n)} - x_i| \leq \epsilon, \tag{$**$}$$

for any $M \geq 1$ and $n \geq N$. We have

$$\begin{aligned}\sum_{i=1}^{M} |x_i| &\leq \sum_{i=1}^{M} |x_i^{(N)} - x_i| + \sum_{i=1}^{M} |x_i^{(N)}| \\ &\leq \epsilon + \|(x_i^{(N)})_{i=1}^{\infty}\|_1.\end{aligned}$$

Letting $M \to +\infty$, we see that $\sum_{i=1}^{\infty} |x_i|$ is finite, so $(x_i)_{i=1}^{\infty} \in \ell^1$.

Finally, letting $M \to +\infty$ in $(**)$ gives

$$\sum_{i=1}^{\infty} |x_i^{(n)} - x_i| \leq \epsilon,$$

for all $n \geq N$, so that $\lim_{n\to+\infty} \|(x_i^{(n)})_{i=1}^{\infty} - (x_i)_{i=1}^{\infty}\|_1 = 0$, as required. $\square$

Theorem 4.10. *For $p \geq 1$, $\ell^p = \left\{(x_i)_{i=0}^{\infty} : \sum_{i=0}^{\infty} |x_i|^p < +\infty,\ x_i \in \mathbb{C}\right\}$ with the norm*

$$\|(x_i)_{i=0}^{\infty}\|_p = \left(\sum_{i=0}^{\infty} |x_i|^p\right)^{1/p}$$

is a Banach space.

Proof. Suppose that $\{(x_i^{(n)})_{i=0}^{\infty}\}_{n=1}^{\infty}$ is a Cauchy sequence for $\|\cdot\|_p$. Then, given $\epsilon > 0$, there exists $N \geq 1$ such that, for $n, m \geq N$ and any $i \in \mathbb{N}$,

$$|x_i^{(n)} - x_i^{(m)}|^p \leq \sum_{i=0}^{\infty} |x_i^{(n)} - x_i^{(m)}|^p = \|(x_i^{(n)})_{i=0}^{\infty} - (x_i^{(m)})_{i=0}^{\infty}\|_p^p < \epsilon^p. \tag{$*$}$$

So, for each fixed i, $\{x_i^{(n)}\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{C}$ and thus has a limit $x_i \in \mathbb{C}$, say.

Next, note that, for any $M \geq 1$ and $n, m \geq N$, we have

$$\sum_{i=0}^{M} |x_i^{(n)} - x_i^{(m)}|^p \leq \sum_{i=0}^{\infty} |x_i^{(n)} - x_i^{(m)}|^p < \epsilon^p.$$

If we let $m \to +\infty$, this gives

$$\sum_{i=0}^{M} |x_i^{(n)} - x_i|^p \leq \epsilon^p, \tag{$**$}$$

for any $M \geq 1$ and $n \geq N$. By Minkowski's inequality,

$$\begin{aligned}\left(\sum_{i=0}^{M}|x_i|^p\right)^{1/p} &\leq \left(\sum_{i=0}^{M}|x_i^{(N)}-x_i|^p\right)^{1/p}+\left(\sum_{i=0}^{M}|x_i^{(N)}|^p\right)^{1/p}\\ &\leq \left(\sum_{i=0}^{M}|x_i^{(N)}-x_i|^p\right)^{1/p}+\left(\sum_{i=0}^{\infty}|x_i^{(N)}|^p\right)^{1/p}\\ &\leq \epsilon + \|(x_i^{(N)})_{i=0}^{\infty}\|_p.\end{aligned}$$

Letting $M \to +\infty$, we see that $\left(\sum_{i=0}^{M}|x_i^{(N)}-x_i|^p\right)^{1/p}$ is finite, so $(x_i)_{i=0}^{\infty} \in \ell^p$.

Finally, letting $M \to +\infty$ in $(**)$ gives

$$\sum_{i=0}^{\infty}|x_i^{(n)}-x_i|^p \leq \epsilon^p,$$

for all $n \geq N$, so that $\lim_{n\to+\infty}\|(x_i^{(n)})_{i=0}^{\infty}-(x_i)_{i=0}^{\infty}\|_p = 0$, as required. □

Example 4.11. $C([0,1],\mathbb{R})$ with the norm $\|f\|_1 = \int_0^1|f(x)|dx$ is *not* a Banach space.

Example 4.12. $(\ell^1, \|\cdot\|_2)$ is not a Banach space. (See exercises.)

4.2 Lebesgue Spaces

As we have seen, the normed space $(C([0,1],\|\cdot\|_1)$ is not a Banach space. Similarly, $(C([0,1],\|\cdot\|_p)$, $1 \leq p < \infty$, are not Banach spaces. However, we can obtain Banach spaces by the process of completion.

Let (X,d) be a metric space which is not complete. There is a complete metric space (X^*,d^*), called the completion of X, such that there is an injective map $\rho : X \to X^*$ with the property that

(a) $\rho(X)$ is dense in X^*;

(b) the restriction of d^* to $X \times X$ is equal to d.

The completion is constructed as follows. Let $\mathcal{X}$ denote the set of all Cauchy sequences in X (with respect to d). We define an equivalence relation on $\mathcal{X}$ by

$$(x_n) \sim (y_n) \qquad \Longleftrightarrow \qquad \lim_{n\to+\infty} d(x_n,y_n) = 0.$$

Then we define $X^* = \mathcal{X}/\sim$ (i.e. X^* is the set of equivalence classes for this relation) and

$$d^*([(x_n)],[(y_n)]) = \lim_{n\to+\infty} d(x_n,y_n),$$

where $[(x_n)] \in X^*$ is the equivalence class of the sequence (x_n). Of course, one needs to show that d^* is well-defined and is a metric; we omit this. Then map $\rho : X \to X^*$ is defined by $\rho(x) = [(x_n)]$, where (x_n) is the constant sequence $x_n = x$ for all n.

The most familiar example of this procedure is to take $X = \mathbb{Q}$ with the standard metric $d(x, y) = |x - y|$. Then $X^* = \mathbb{R}$.

More exotic completions of $\mathbb{Q}$ are the p-adic numbers $\mathbb{Q}_p$. Let p be a prime number. For $x \in \mathbb{Q}$, $x \neq 0$, there is a unique $n \in \mathbb{Z}$ such that

$$x = p^n \frac{a}{b},$$

where a and b are integers which are not divisible by p. We define $|0|_p = 0$ and, for $x \neq 0$,

$$|x|_p = p^{-n}.$$

One can show that $d(x, y) = |x - y|_p$ is a metric on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to this metric (which happens also to be a field) is called the p-adic numbers, written $\mathbb{Q}_p$.

Remark 4.13. It is important not to confuse the completion of a set with its closure. It so happens that the closure of $\mathbb{Q}$ in $\mathbb{R}$ is equal to $\mathbb{R}$ but in the metric space $(\mathbb{Q}, |\cdot - \cdot|)$, $\mathbb{Q}$ is already closed.

The completion of $C([a, b], \mathbb{R})$ with respect to the norm $\|\cdot\|_p$, $1 \leq p < \infty$, is denoted $L^p([a, b], \mathbb{R})$. (The real number p here has no connection with the prime p above!) These spaces are called Lebesgue spaces. When speaking, to distinguish them from the spaces ℓ^p, we call them "big L^p" and "little ℓ^p".

The spaces $L^p([a, b], \mathbb{R})$ have a separate definition involving measure theory.

Definition 4.14. A collection $\mathcal{B}$ of subsets of a set X is called a *σ-algebra* if

(i) $\varnothing \in \mathcal{B}$

(ii) $\mathcal{B}$ is closed under complements: $E \in \mathcal{B} \implies X \backslash E \in \mathcal{B}$

(iii) $\mathcal{B}$ is closed under *countable* unions: $\{E_j\}_{j=1}^{\infty} \subset \mathcal{B} \implies \bigcup_{j=1}^{\infty} E_j \in \mathcal{B}$.

(If, in (iii), we replaced "countable" by "finite" then $\mathcal{B}$ would just be called an algebra.)

It follows from the definition that if $\mathcal{B}$ is a σ-algebra then $X \in \mathcal{B}$ and $\mathcal{B}$ is closed under countable intersections: $\{E_j\}_{j=1}^{\infty} \subset \mathcal{B} \implies \bigcap_{j=1}^{\infty} E_j \in \mathcal{B}$.

Simple examples of σ-algebras are $\mathcal{B} = \{\varnothing, X\}$ and $\mathcal{B} = \mathcal{P}(X)$ (the set of all subsets of X). A more important σ-algebra is the following.

Definition 4.15. Let X be a metric space and let $\mathcal{O}$ denote the collection of all open sets in X. The smallest σ-algebra containing $\mathcal{O}$ is called the *Borel σ-algebra* of X.

Now let us specialise to the set $\mathbb{R}$. The object we are interested in, denoted below by μ and called Lebesgue measure, is defined on a σ-algebra $\mathcal{M}$ of subsets of $\mathbb{R}$ which contains the Borel σ-algebra. Sets in $\mathcal{M}$ are called *measurable sets*.

Theorem 4.16 (Existence of Lebesgue measure)**.** *There exists a unique function $\mu : \mathcal{M} \to \mathbb{R}^+ \cup \{+\infty\}$ such that*

(i) if I is an interval then $\mu(I) = \text{length}(I)$ (length property);

(ii) for every $x \in \mathbb{R}$, $\mu(x + E) = \mu(E)$ (translation invariance);

(iii) if $\{E_j\}_{j=1}^{\infty}$ is a countable collection of disjoint sets (in $\mathcal{M}$) then

$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j)$$

(countable additivity);

(iv) $E \subset F \implies \mu(E) \leq \mu(F)$ (monotonicity);

(v) if $A \in \mathcal{M}$ then $\mu(A) = \inf\{\mu(U) : U$ is open and $A \subset U\}$ (regularity).

The Lebesgue measure may be used to define integrals in the following way. We will define integrals over an interval $[a, b]$ (though we could do this for any subset of $\mathbb{R}$) and work with the restriction $\mu : \mathcal{M}([a, b]) \to \mathbb{R}$, where $\mathcal{M}([a, b]) = \{A \cap [a, b] : A \in \mathcal{M}\}$. (Note that μ is finite valued on $\mathcal{M}([a, b])$).

A function $f : [a, b] \to \mathbb{R} \cup \{\pm\infty\}$ is called *measurable* if $f^{-1}((c, +\infty) \in \mathcal{M}([a, b])$ for every c. (There are lots of equivalent formulations.) Note that continuous functions are measurable. A measurable function f is called *simple* if it takes only finitely many values, $a_1, \ldots, a_n$ say, and we define its integral to be

$$\int_a^b f(x)\, dx = \sum_{i=1}^{n} a_i \mu(E_i),$$

where $E_i = f^{-1}(\{a_i\})$. (If we want to emphasise the measure, we might write $\int_a^b f\, d\mu$.)

This definition can be extended using the following lemma.

Lemma 4.17. *Let $f : [a, b] \to \mathbb{R} \cup +\infty$ be a non-negative measurable function. Then there exists an increasing sequence of non-negative simple functions f_n, $n \geq 1$, such that f_n converges pointwise to f, as $n \to +\infty$.*

We then define

$$\int_a^b f(x)\, dx = \lim_{n \to +\infty} \int_a^b f_n(x)\, dx.$$

(One can show that this is well-defined.) We can extend this to functions f which take both positive and negative values provided $\int_a^b f^+(x)\, dx$ and $\int_a^b f^-(x)\, dx$ are both finite, where $f^+(x) = \max\{f(x), 0\}$ and $f^-(x) = \max\{-f(x), 0\}$. This condition is equivalent to $\int_a^b |f(x)|\, dx < +\infty$ and we called such functions *integrable* (over $[a, b]$). If f is integrable, we define

$$\int_a^b f(x)\, dx = \int_a^b f^+(x)\, dx - \int_a^b f^-(x)\, dx.$$

These definitions can be extended to complex valued functions by considering real and imaginary parts.

Two measurable functions f and g are said to be equivalent if $f = g$ μ-almost everywhere. This means that $\mu(\{x \in [a, b] : f(x) \neq g(x)\}) = 0$. Whether f is integrable and the value of the integral only depends on the equivalence class $[f]$ of f. We define $L^1([a, b], \mathbb{R})$ to be the set of equivalence classes of integrable functions $f : [a, b] \to \mathbb{R}$. More generally, for $1 \leq p < +\infty$,

$$L^p([a, b], \mathbb{R}) = \left\{ [f] : f \text{ measurable}, \int_a^b |f(x)|^p \, dx < +\infty \right\}.$$

We have the following theorem, which we do not prove.

Theorem 4.18. *The two definitions of $L^p([a, b], \mathbb{R})$ are equivalent. In particular, $L^p([a, b], \mathbb{R})$ is a Banach space with respect to the norm $\| \cdot \|_p$.*

If $1 \leq p < q < +\infty$ then $L^q([a, b], \mathbb{R}) \subset L^p([a, b], \mathbb{R})$ but $L^q([a, b], \mathbb{R}) \neq L^p([a, b], \mathbb{R})$. (Notice that these inclusions are the reverse of those for ℓ^p.)

There is another Banach space we can define in this context. For a measurable function $f : [a, b] \to \mathbb{R} \cup \{\pm\infty\}$, define

$$\|f\|_\infty = \inf \left\{ \sup_{y \in Y} |f(y)| : Y \subset [a, b], \ \mu([a, b] \backslash Y) = 0 \right\}.$$

(If f is continuous then this agrees with the supremum norm of f.) Define

$$L^\infty([a, b], \mathbb{R}) = \{[f] : f \text{ measurable}, \ \|f\|_\infty < +\infty\}.$$

This is a Banach space with respect to $\|\cdot\|_\infty$. One can show that $L^\infty([a, b], \mathbb{R}) \subset L^1([a, b], \mathbb{R})$ but $L^\infty([a, b], \mathbb{R}) \neq L^1([a, b], \mathbb{R})$.

These constructions can be considerably generalised. Let $(X, \mathcal{B})$ be a set and a σ-algebra of its subsets. In this setting a *measure* is any function $\mu : \mathcal{B} \to \mathbb{R}^+ \cup \{+\infty\}$ such that $\mu(\varnothing) = 0$ and, if $\{E_j\}_{j=1}^\infty$ is a countable collection of disjoint sets in $\mathcal{B}$ then

$$\mu\left(\bigcup_{j=1}^\infty E_j \right) = \sum_{j=1}^\infty \mu(E_j).$$

If $\mu(X) < +\infty$, μ is called a finite measure. One can then define, using the construction above, measurable and integrable functions $f : X \to \mathbb{R} \cup \{\pm\infty\}$ and an integral which we will denote $\int_X f \, d\mu$. The spaces $L^p(X, \mu, \mathbb{R})$ are defined in the way one would expect and are Banach spaces with respect to

$$\|f\|_p = \left(\int_X |f|^p \, d\mu \right)^{1/p}.$$

Similarly, one may define a Banach space $L^\infty(X, \mu, \mathbb{R})$. If μ is a finite measure the inclusions above hold. However, if $\mu(X) = +\infty$ they fail to hold. Indeed, if $X = \mathbb{R}$ with Lebesgue measure then, for $p \neq q$, $L^p(\mathbb{R}, \mu, \mathbb{R}) \not\subset L^q(\mathbb{R}, \mu, \mathbb{R})$ and $L^q(\mathbb{R}, \mu, \mathbb{R}) \not\subset L^p(\mathbb{R}, \mu, \mathbb{R})$.

Now suppose X is a compact metric space, that $\mathcal{B}$ contains the Borel σ-algebra and that μ is a finite measure. It is not necessarily the case that $\|\cdot\|_p$ defines a norm on $C(X, \mathbb{R})$. For example, if X contains more than one point, take $x_0 \in X$ and define μ by $\mu(A) = 1$ if $x_0 \in A$ and $\mu(A) = 0$ otherwise. (It is easy to check this is a measure with $\mathcal{B} = \mathcal{P}(X)$.) There are continuous functions $f \neq g$ on X with $f(x_0) = g(x_0)$ but

$$\|f - g\|_p = \left(\int_X |f - g|^p \, d\mu \right)^{1/p} = (|f(x_0) - g(x_0)|^p)^{1/p} = |f(x_0) - g(x_0)| = 0.$$

However, if μ has the property that $\mu(U) > 0$ for every open set $U \subset X$ (this is called being fully supported) then $\|\cdot\|_p$ is a norm on $C(X, \mathbb{R})$ and, furthermore, its completion is $L^p(X, \mu, \mathbb{R})$. (Note that it is important for X to be compact here as, for non-compact X, there will often be continuous functions for which $\|f\|_p$ is infinite.

Remark 4.19. In fact, the little ℓ spaces fit into this setting. Let $X = \mathbb{N} \cup \{0\}$ and let μ be the counting measure on X: $\mu(A)$ is equal to the the cardinality of A if A if finite and $+\infty$ if A is infinite. Then

$$\ell^p(\mathbb{R}) = L^p(X, \mu, \mathbb{R}).$$

We end the section by noting the following theorem about the convergence of the integrals defined above. It is valid for any measure.

Theorem 4.20 (Dominated Convergence Theorem)**.** *Let f_n, $n \geq 1$, be a sequence of integrable functions such that $f_n(x)$ converges to $f(x)$ for μ-a.e. x. Suppose there exists an integrable function $g \geq 0$ such that, for all $n \geq 1$, $|f_n(x)| \leq g(x)$ for μ-a.e x. Then f is integrable and*

$$\lim_{n \to +\infty} \int_a^b f_n(x) \, dx = \int_a^b f(x) \, dx.$$

5 Hilbert Spaces

5.1 Inner Products

Hilbert spaces are special types of Banach spaces in which the norm is defined by an inner product. A particular feature of these spaces is that one has the notion of orthogonality. They are named after David Hilbert (1862-1943): he and Poincaré were the most important mathematicians of the early 20th century.

Definition 5.1. Let H be a vector space over $\mathbb{R}$ or $\mathbb{C}$. An *inner product* is a map $\langle\cdot,\cdot\rangle : H \times H \to \mathbb{R}$ (or $\mathbb{C}$) such that, for all $x, y, z \in H$ and scalars λ, μ,

(1) $\langle x, y\rangle = \overline{\langle y, x\rangle}$ (complex conjugation);

(2) $\langle \lambda x + \mu y, z\rangle = \lambda\langle x, z\rangle + \mu\langle y, z\rangle$; and

(3) $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0$ if and only if $x = 0$.

Of course, if the vector space is over $\mathbb{R}$ then (1) is just $\langle x, y\rangle = \langle y, x\rangle$.

This definition generalizes the familiar inner product (= dot product) on $\mathbb{R}^n$.

Example 5.2. (a) $H = \mathbb{R}^n$, $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$ is an inner product.

(b) $H = \mathbb{C}^n$, $\langle x, y\rangle = \sum_{i=1}^n x_i \bar{y}_i$ is an inner product.

With the above example in mind, the following result is a more abstract version of the Cauchy-Schwarz inequality.

Lemma 5.3 (Cauchy-Schwarz Inequality for Inner Products)**.** *Let $\langle\cdot,\cdot\rangle$ be an inner product on H. Then*

$$|\langle x, y\rangle| \leq \langle x, x\rangle^{1/2}\langle y, y\rangle^{1/2},$$

for all $x, y \in H$.

Proof. From the definition of inner product, for any λ,

$$\begin{aligned} 0 \leq \langle x - \lambda y, x - \lambda y\rangle &= \langle x, x\rangle - 2\Re(\lambda\langle x, y\rangle) + |\lambda|^2\langle y, y\rangle \\ &= (\langle x, x\rangle^{1/2} - |\lambda|\langle y, y\rangle^{1/2})^2 - 2\Re(\lambda\langle x, y\rangle) + 2|\lambda|\langle x, x\rangle^{1/2}\langle y, y\rangle^{1/2}. \end{aligned}$$

If $y = 0$ the result is clear, so we suppose $y \neq 0$ and let

$$\lambda = \frac{\langle x, x\rangle^{1/2}}{\langle y, y\rangle^{1/2}} e^{i\theta},$$

where $\langle x, y\rangle = e^{-i\theta}|\langle x, y\rangle|$. Substituting, we get

$$\begin{aligned} 0 &\leq 0 - 2\Re\left(\frac{\langle x, x\rangle^{1/2}}{\langle y, y\rangle^{1/2}} e^{i\theta}\langle x, y\rangle\right) + 2\frac{\langle x, x\rangle^{1/2}}{\langle y, y\rangle^{1/2}}\langle x, x\rangle^{1/2}\langle y, y\rangle^{1/2} \\ &= -2\left(\frac{\langle x, x\rangle^{1/2}}{\langle y, y\rangle^{1/2}}|\langle x, y\rangle|\right) + 2\langle x, x\rangle. \end{aligned}$$

Rearranging, we get the required inequality. $\square$

Lemma 5.4. *Suppose that $\langle\cdot,\cdot\rangle$ is an inner product on H. Then $\|\cdot\|$ defined by $\|x\| = \langle x, x\rangle^{1/2}$ is a norm on H.*

Proof. It is immediate from the definition of inner product that $\|x\| \geq 0$ and $\|x\| = 0$ iff and only if $x = 0$. We also have

$$\|\lambda x\| = \langle \lambda x, \lambda x\rangle^{1/2} = \left(\lambda\bar{\lambda}\langle x, x\rangle\right)^{1/2} = |\lambda|\langle x, x\rangle^{1/2} = |\lambda|\|x\|.$$

Finally, we need to show that $\|x + y\| \leq \|x\| + \|y\|$. We have

$$\begin{aligned}\|x+y\|^2 = \langle x+y, x+y\rangle &= \langle x, x\rangle + 2\Re\langle x, y\rangle + \langle y, y\rangle \\ &\leq \langle x, x\rangle + 2\langle x, x\rangle^{1/2}\langle y, y\rangle^{1/2} + \langle y, y\rangle \\ &= (\langle x, x\rangle^{1/2} + \langle y, y\rangle^{1/2})^2 = (\|x\| + \|y\|)^2\end{aligned}$$

(using the Cauchy-Schwarz inequality and $\Re z \leq |z|$), as required. □

In view of this, we may restate the Cauchy-Schwarz inequality in a more convenient form.

Lemma 5.5 (Restatement of the Cauchy-Schwarz Inequality for Inner Products)**.** *Let $\langle\cdot,\cdot\rangle$ be an inner product on H and let $\|\cdot\|$ be the associated norm. Then*

$$|\langle x, y\rangle| \leq \|x\|\|y\|,$$

for all $x, y \in H$.

Corollary 5.6. *The map $\langle\cdot,\cdot\rangle : H \times H \to \mathbb{C}$ is continuous (with respect to the associated norm).*

Proof. Suppose that $(x_n, y_n) \to (x, y)$ in $H \times H$, as $n \to +\infty$. Then $\lim_{n\to+\infty} \|x_n - x\| = 0$ and $\lim_{n\to+\infty} \|y_n - y\| = 0$. Using the Cauchy-Schwarz inequality,

$$\begin{aligned}|\langle x_n, y_n\rangle - \langle x, y\rangle| &= |\langle x_n, y_n\rangle - \langle x_n, y\rangle + \langle x_n, y\rangle - \langle x, y\rangle| \\ &= |\langle x_n, y_n - y\rangle + \langle x_n - x, y\rangle| \\ &\leq \|x_n\|\|y_n - y\| + \|x_n - x\|\|y\| \to 0, \text{ as } n \to +\infty\end{aligned}$$

(since x_n converges, $\|x_n\|$ is bounded). □

Now, at last, we can give the definition of a Hilbert space.

Definition 5.7. A *Hilbert space* is a vector space H with an inner product $\langle\cdot,\cdot\rangle$ such that H is complete with respect to the associated norm $\|x\| = \langle x, x\rangle^{1/2}$.

Another way of putting this is that a Hilbert space is a Banach space where the norm is given by an inner product.

Example 5.8. $\mathbb{R}^n$ is a Hilbert space with inner product

$$\langle (x_1, \ldots, x_n), (y_1, \ldots, y_n)\rangle = x_1 y_1 + \cdots + x_n y_n$$

and $\mathbb{C}^n$ is a Hilbert space with inner product

$$\langle (x_1, \ldots, x_n), (y_1, \ldots, y_n)\rangle = x_1\overline{y_1} + \cdots + x_n\overline{y_n}.$$

Example 5.9. ℓ^2 is a Hilbert space, with the inner product given by

$$\langle (x_i)_{i=0}^\infty, (y_i)_{i=0}^\infty \rangle = \sum_{i=0}^\infty x_i \bar{y}_i.$$

(Check that $\langle (x_i)_{i=0}^\infty, (x_i)_{i=0}^\infty \rangle^{1/2} = \|(x_i)_{i=0}^\infty\|_2$.) We use the Cauchy-Schwarz inequality to ensure that the inner product is finite.

Example 5.10. $L^2([a,b],\mathbb{R})$ is a Hilbert space with inner product

$$\langle f, g \rangle = \int_0^1 f(x)g(x)\,dx.$$

$L^2([a,b],\mathbb{C})$ is a Hilbert space with inner product

$$\langle f, g \rangle = \int_0^1 f(x)\overline{g(x)}\,dx.$$

Example 5.11. Let $H = M_n(\mathbb{C}) = n \times n$ complex matrices. This is a Hilbert space with respect to the inner product $\langle A, B \rangle = \text{trace}(AB^*)$, where B^* is the matrix with (i,j) entry $B^*(i,j) = \overline{B(j,i)}$.

The following is a useful property of Hilbert spaces.

Lemma 5.12 (Parallelogram law)**.** *Let H be a Hilbert space. For $x, y \in H$, we have the identity*

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

Proof. We have the identities

$$\|x+y\|^2 = \langle x+y, x+y \rangle = \|x\|^2 + \langle x, y \rangle + \langle y, x \rangle + \|y\|^2$$

and

$$\|x-y\|^2 = \langle x-y, x-y \rangle = \|x\|^2 - \langle x, y \rangle - \langle y, x \rangle + \|y\|^2.$$

Adding them together gives the result. □

5.2 Closest approximation in convex sets

Definition 5.13. A subset C of a vector space V is convex if, for all $x, y \in C$, we have $\lambda x + (1-\lambda)y \in C$ for all $\lambda \in [0,1]$.

Lemma 5.14. *If C is a non-empty closed convex subset of a Hilbert space H then, for each $x \in H$, there is a unique $a^* \in C$ such that*

$$\|x - a^*\| = \inf_{a \in C} \|x - a\|.$$

Proof. For $x, u, v \in H$, the parallelogram law implies that

$$\|(x-u)+(x-v)\|^2 + \|(x-u)-(x-v)\|^2 = 2\|x-u\|^2 + 2\|x-v\|^2.$$

Then (since $\|(x-u)-(x-v)\| = \|u-v\|$)

$$\|u-v\|^2 = 2\|x-u\|^2 + 2\|x-v\|^2 - 4\left\|x - \frac{1}{2}(u+v)\right\|^2.$$

Let $d = \inf_{a\in C}\|x-a\|$. Now suppose $u, v \in C$. Since C is convex, $(u+v)/2 \in C$ and, consequently, $\|x-(u+v)/2\| \geq d$. Then

$$\|u-v\|^2 \leq 2\|x-u\|^2 + 2\|x-v\|^2 - 4d^2. \tag{$*$}$$

Since d is the infimum, for any $n \geq 1$, there exists $a_n \in C$ such that

$$\|x-a_n\|^2 < d^2 + \frac{1}{n}.$$

Using (*) with $u = a_n$ and $v = a_m$, we obtain

$$\|a_n - a_m\|^2 \leq 2d^2 + \frac{2}{n} + 2d^2 + \frac{2}{m} - 4d^2 = \frac{2}{n} + \frac{2}{m}.$$

Hence a_n is a Cauchy sequence and so, since H is complete, converges to some $a^* \in H$. Since C is closed, $a^* \in C$. We will show that a^* is the required point. First,

$$\|x-a^*\|^2 = \lim_{n\to+\infty}\|x-a_n\|^2 = d^2,$$

so $\|x-a^*\|$ is equal to the desired infimum. Now suppose that there is another point $a' \in C$ with $\|x-a'\| = d$. Then using (*) with $u = a^*$ and $v = a'$ gives

$$\|a^*-a'\|^2 \leq 2\|x-a^*\|^2 + 2\|x-a'\|^2 - 4d^2 = 2d^2 + 2d^2 - 4d^2 = 0.$$

Hence $a' = a^*$ and a^* is unique. □

5.3 Orthogonal Complements

Let H be a Hilbert space over $\mathbb{R}$ (or $\mathbb{C}$) and let L be a linear subspace of H. Then L is a convex set, so the result of the preceding subsection applies.

Definition 5.15. The *orthogonal complement* $L^\perp$ of L is defined by

$$L^\perp = \{x \in H : \langle x, y\rangle = 0 \text{ for all } y \in L\}.$$

Lemma 5.16. *Let H be a Hilbert space and let L be a linear subspace. Then $L^\perp$ is a closed linear subspace of H and $H = \overline{L} \oplus L^\perp$. (Here $\overline{L}$ denotes the closure of L.)*

Proof. First we show that $L^\perp$ is linear. Suppose that $x, y \in L^\perp$, λ, μ are scalars and that $z \in L$. Then
$$\langle \lambda x + \mu y, z\rangle = \lambda\langle x, z\rangle + \mu\langle y, z\rangle = 0 + 0 = 0,$$
so $\lambda x + \mu y \in L^\perp$.

Next we show that $L^\perp$ is closed. Suppose that $x_n \in L^\perp$ and that $x_n \to x$ in H, as $n \to +\infty$. For any $z \in L$, we have
$$\begin{aligned}|\langle x, z\rangle| &= |\langle x, z\rangle - \langle x_n, z\rangle| = |\langle x - x_n, z\rangle| \\ &\leq \|x - x_n\|\|z\| \to 0, \text{ as } n \to +\infty.\end{aligned}$$
thus $\langle x, z\rangle = 0$, for all $z \in L$, i.e., $x \in L^\perp$.

Finally, we show that $H = \overline{L} \oplus L^\perp$. Recall that this means that any $x \in H$ has a unique decomposition $x = x_1 + x_2$, where $x_1 \in \overline{L}$ and $x_2 \in L^\perp$. Since $\overline{L}$ is a linear space it is convex and by definition it is closed, so we can apply Lemma 5.14. For $x \in H$ let x_1 be the unique point in $\overline{L}$ which is closest to x. To complete the proof, we claim that $x_2 := x - x_1 \in L^\perp$. Assume that this is false, then there exists $v \in L$ with $\langle x - x_1, v\rangle \neq 0$. By multiplying v by a scalar $e^{i\theta}$, we may assume that $\langle x - x_1, e^{i\theta}v\rangle = -c < 0$. (If H is over $\mathbb{R}$ then $\theta = 0$ or π, so $e^{i\theta}v = v$ or $-v$.) Thus, if $t > 0$ is sufficiently small,
$$\begin{aligned}\|x - x_1 + te^{i\theta}v\|^2 &= \|x - x_1\|^2 + t^2\|v\|^2 + (\langle x - x_1, te^{i\theta}v\rangle + \langle te^{i\theta}v, x - x_1\rangle) \\ &= \|x - x_1\|^2 + t^2\|v\|^2 - 2tc < \|x - x_1\|^2,\end{aligned}$$
i.e. $\|x - x_1 + te^{i\theta}v\| < \|x - x_1\| = \inf\{\|x - u\| : u \in L\}$, a contradiction. Therefore $x_2 = x - x_1 \in L^\perp$ and we are finished. □

Corollary 5.17. *If L is a closed linear subspace of H then $H = L \oplus L^\perp$.*

Proof. If L is closed then $\overline{L} = L$. □

Let L be a closed linear subspace of H. We will define a map $P_L : H \to L$ by $P_L(x) = x_1$, where x_1 is the point in L which is closest to x as defined in the proof of Lemma 5.16. We call P_L the orthogonal projection from H to L.

Lemma 5.18. *P_L is a linear map, $P_L^2 = P_L$ and $\|P_L(x)\| \leq \|x\|$.*

Proof. For scalars λ and μ,
$$\lambda x + \mu y - (\lambda P_L(x) + \mu P_L(y)) = \lambda(x - P_L(x)) + \mu(y - P_L(y)) \in L^\perp,$$
so that $P_L(\lambda x + \mu y) = \lambda P_L(x) + \mu P_L(y)$, i.e. P_L is linear. If $x \in L$ then $P_L(x) = x$, so it follows that $P_L^2 = P_L$. Finally, $x = P_L(x) + y$ with $y \in L^\perp$. Since $\langle P_L(x), y\rangle = 0$, $\|x\|^2 = \|P_L(x)\|^2 + \|y\|^2$, so $\|P_L(x)\| \leq \|x\|$. □

5.4 Orthonormal sets and best approximations

Definition 5.19. A subset $\{e_n\}_{n=1}^\infty$ of a Hilbert space is called *orthonormal* if

$$\langle e_i, e_j\rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

In other words $\langle e_i, e_j\rangle = 0$ if $i \neq j$ and $\|e_i\| = 1$.

Let $\{f_n\}_{n=1}^\infty \subset H$ be a linearly independent subset. Then, using a procedure called the *Gram-Schmidt algorithm*, it is possible to construct an orthonormal set $\{e_n\}_{n=1}^\infty \subset H$ which has the same span. We first let $e_1 = f_1/\|f_1\|$. Next we set

$$g_2 = f_2 - \langle f_2, e_1\rangle e_1$$

and let $e_2 = g_2/\|g_2\|$. Then we set

$$g_3 = f_3 - \langle f_3, e_1\rangle e_1 - \langle f_3, e_2\rangle e_2$$

and let $e_3 = g_3/\|g_3\|$. We continue in a similar fashion to define the whole sequence $\{e_n\}_{n=1}^\infty$.

The following inequality is useful.

Proposition 5.20 (Bessel's inequality)**.** *Let $\{e_n\}_{n=1}^\infty \subset H$ be an orthonormal set. Let $x \in H$ and write $a_n = \langle x, e_n\rangle$, $n \geq 1$. Then, for all $N \geq 1$,*

$$\sum_{n=1}^{N} |a_n|^2 \leq \|x\|^2.$$

In particular, $\sum_{n=1}^\infty |a_n|^2 \leq \|x\|^2$.

Proof. We may write

$$\begin{aligned} 0 &\leq \left\| x - \sum_{n=1}^{N} a_n e_n \right\|^2 \\ &= \langle x, x\rangle - \left\langle x, \sum_{n=1}^{N} a_n e_n \right\rangle - \left\langle \sum_{n=1}^{N} a_n e_n, x \right\rangle + \left\langle \sum_{n=1}^{N} a_n e_n, \sum_{n=1}^{N} a_n e_n \right\rangle \\ &= \|x\|^2 - \sum_{n=1}^{N} a_n \overline{a}_n - \sum_{n=1}^{N} a_n \overline{a}_n + \sum_{n=1}^{N} a_n \overline{a}_n \\ &= \|x\|^2 - \sum_{n=1}^{N} a_n \overline{a}_n. \end{aligned}$$

This gives the required inequality. □

There is a nice relationship between orthonormal bases and finding points closest to a given point.

Theorem 5.21. *Let $\{e_n\}$ be an orthonormal subset of a Hilbert space H. Then for any $x \in H$ the closest point to x in $\overline{\operatorname{span}(\{e_n\})}$ is*

$$\sum_{n=1}^{\infty} \langle x, e_n \rangle e_n.$$

Proof. By Bessel's inequality,

$$\sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2 \leq \|x\|^2,$$

so the sum $\sum_{n=1}^{\infty} \langle x, e_n \rangle e_n$ converges in H. Clearly, the limit is in $\overline{\operatorname{span}(\{e_n\})}$. Let

$$y = x - \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n.$$

Since, for $k \geq 1$

$$\langle y, e_k \rangle = \langle x, e_k \rangle - \langle x, e_k \rangle = 0,$$

we have $y \in (\overline{\operatorname{span}(\{e_n\})})^{\perp}$. Therefore, by Lemma 5.16, $\sum_{n=1}^{\infty} \langle x, e_n \rangle e_n$ is the desired closest point. □

Corollary 5.22. *Let L be a closed linear subspace of H and let $\{e_n\} \subset L$ be an orthonormal set with $\overline{\operatorname{span}(\{e_n\})} = L$. Then*

$$P_L(x) = \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n.$$

5.5 Separable Hilbert Spaces

A metric space is called separable if it contains a countable dense subset. Finite dimensional normed vector spaces are separable. For any compact metric space X, $C(X, \mathbb{R})$ and $C(X, \mathbb{C})$ are separable (this is a consequence of the Stone-Weierstrass Theorem) and, for $1 \leq p < +\infty$, ℓ^p and $L^p([a, b], \mathbb{R})$ are separable. However, ℓ^∞ and $L^\infty([a, b], \mathbb{R})$ are not separable.

A key concept will be that of complete orthonormal sets.

Definition 5.23. A set $\{e_n\}_{n=1}^{\infty} \subset H$ is a complete orthonormal set if

(1) it is orthonormal (see above); and

(2) it is complete, in the sense that if $x \in H$ has $\langle x, e_n \rangle = 0$, for all $n \geq 1$, then $x = 0$. (This is *different* from the notion of completeness in metric spaces.)

We shall show that every separable Hilbert space contains a complete orthonormal set. The first step is to show that *if* $\{e_n\}_{n=1}^{\infty}$ is a complete orthonormal set then any $x \in H$ may be written in terms of the e_n.

Proposition 5.24. *Let $\{e_n\}_{n=1}^{\infty} \subset H$ be a complete orthonormal set. If $x \in H$ and $a_n = \langle x, e_n \rangle$ then*

$$x = \lim_{N \to +\infty} \sum_{n=1}^{N} a_n e_n = \sum_{n=1}^{\infty} a_n e_n.$$

Proof. Let $w_N = \sum_{n=1}^{N} a_n e_n$. Then, for $N \geq M$,

$$\|w_N - w_M\|^2 = \left\| \sum_{n=M+1}^{N} a_n e_n \right\|^2 = \sum_{n=M+1}^{N} |a_n|^2.$$

By Bessel's inequality, $\sum_{n=M+1}^{N} |a_n|^2 \to 0$, as $N, M \to +\infty$, so $\{w_N\}$ is a Cauchy sequence. Hence, w_N converges to some $w \in H$, as $N \to +\infty$. In other words, $w = \sum_{n=1}^{\infty} a_n e_n$.

Now, for all $n \geq 1$, we have

$$\langle w, e_n \rangle = a_n = \langle x, e_n \rangle.$$

Thus $\langle w - x, e_n \rangle = 0$. By the completeness of $\{e_n\}_{n=1}^{\infty}$, this means that $w - x = 0$, i.e., $w = x$. $\square$

Bessel's inequality shows that the sequence $\{\langle x, e_n \rangle\}_{n=1}^{\infty}$ is an element of ℓ^2. The next result gives a converse: if $(a_n)_{n=1}^{\infty}$ is any sequence in ℓ^2 then $\sum_{n=1}^{\infty} a_n e_n$ gives an element of H:

Theorem 5.25 (Riesz-Fischer Theorem)**.** *Let $\{e_n\}_{n=1}^{\infty} \subset H$ be an orthonormal set (it doesn't need to be complete) and let $(a_n)_{n=1}^{\infty}$ be an arbitrary sequence in ℓ^2. Then there exists $x \in H$ such that $a_n = \langle x, e_n \rangle$ and $\|(a_n)_{n=1}^{\infty}\|_2 = \|x\|$. (If, in addition, $\{e_n\}_{n=1}^{\infty}$ is complete then, by Proposition 5.24, $x = \sum_{n=1}^{\infty} a_n e_n$.)*

Proof. Suppose $m < p$. Since $(a_n)_{n=1}^{\infty} \in \ell^2$, we have $\left\|\sum_{n=m}^{p} a_n e_n\right\|^2 = \sum_{n=m}^{p} |a_n|^2 \to 0$, as $m, p \to +\infty$, i.e., $\sum_{n=1}^{N} a_n e_n$ is a Cauchy sequence in H; hence it converges to some $x \in H$. Now if $N > k$,

$$\left\langle x - \sum_{n=1}^{N} a_n e_n, e_k \right\rangle = \langle x, e_k \rangle - a_k.$$

By the Cauchy-Schwarz inequality (noting that $\|e_k\| = 1$),

$$|\langle x, e_k \rangle - a_k| \leq \left\| x - \sum_{n=1}^{N} a_n e_n \right\| \to 0, \text{ as } N \to +\infty.$$

Therefore $a_n = \langle x, e_n \rangle$.

Finally,

$$\sum_{n=1}^{N} |a_n|^2 = \left\| \sum_{n=1}^{N} a_n e_n \right\|^2$$

and $\lim_{N\to+\infty}\left\|\sum_{n=1}^{N} a_n e_n\right\| = \|x\|$. Therefore, letting $N \to +\infty$, we have

$$\sum_{n=1}^{\infty} |a_n|^2 = \|x\|^2,$$

i.e., $\|(a_n)_{n=1}^{\infty}\|_2 = \|x\|$. □

We now show that complete orthonormal sets always exist.

Proposition 5.26. *If $\{f_n\}_{n=1}^{\infty}$ is a linearly independent subset of H such that $\operatorname{span}(\{f_n\}_{n=1}^{\infty})$ is dense in H then the orthonormal set $\{e_n\}_{n=1}^{\infty}$ constructed by the Gram-Schmidt algorithm is complete.*

Proof. Let $E = \operatorname{span}(\{f_n\}_{n=1}^{\infty}) = \operatorname{span}(\{e_n\}_{n=1}^{\infty})$. Since E is dense, $E^{\perp} = \{0\}$. If $\langle x, e_n\rangle = 0$, for all $n \geq 1$, then $\langle x, e\rangle = 0$, for all $e \in E$. Thus $x \in E^{\perp}$, so $x = 0$. This shows that $\{e_n\}_{n=1}^{\infty}$ is complete. □

Proposition 5.27. *Every (infinite dimensional) separable Hilbert space contains a complete orthonormal set $\{e_n\}_{n=1}^{\infty}$. Furthermore, each $x \in H$ can be written uniquely in the form*

$$x = \sum_{n=1}^{\infty} a_n e_n, \quad \textit{where } a_n = \langle x, e_n\rangle.$$

Proof. Since H is separable it contains a countable dense subset $\{g_n\}_{n=1}^{\infty}$ and we have $\operatorname{span}(\{g_n\}_{n=1}^{\infty}) = H$. By deleting an element g_n if it is a liners combination of $g_1, \ldots, g_{n-1}$, we can find a linearly independent subset $\{f_n\}_{n=1}^{\infty}$ such that $\operatorname{span}(\{f_n\}_{n=1}^{\infty}) = H$. By Proposition 5.26, we can use the Gram-Schmidt algorithm to construct a complete orthonormal set $\{e_n\}_{n=1}^{\infty}$. By Proposition 5.24, any $x \in H$ can be represented as $x = \sum_{n=1}^{\infty}\langle x, e_n\rangle e_n$. □

Example 5.28. Let $H = \ell^2 = \{(a_i)_{i=0}^{\infty} : \sum_{i=0}^{\infty} |a_i|^2 < +\infty\}$ and $\langle (a_i)_{i=0}^{\infty}, (b_i)_{i=0}^{\infty}\rangle = \sum_{i=0}^{\infty} a_i \overline{b}_i$.
Let $e_n = (\underbrace{0, \ldots, 0}_{\times(n-1)}, 1, 0, \ldots)$ then

(1) $\{e_n\}$ are orthonormal:

$$\langle e_n, e_m\rangle = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases};$$

(2) $\{e_n\}$ is complete: $\langle (a_i)_{i=0}^{\infty}, e_n\rangle = a_n = 0\ \forall n \geq 0 \implies (a_i)_{i=0}^{\infty} = 0$.

Definition 5.29. A countable subset $\{e_n\}$ of a normed vector space $(V, \|\cdot\|)$ is called a Schauder basis if it is linearly independent and if, given $x \in V$, there exist scalars $\{a_n\}$ such that

$$x = \sum_{n=1}^{\infty} a_n e_n$$

(i.e. $\lim_{N\to+\infty}\|x - \sum_{n=1}^{N} a_n e_n\| = 0$.) Note that this differs from a Hamel basis by (a) being countable but (b) allowing elements to be represented by infinite linear combinations. If we are in a Hilbert space, it is conventional just to refer to a Schauder basis as a basis.

Lemma 5.30. *A complete orthonormal subset of an infinite dimensional separable Hilbert space is a basis.*

Proof. This follows immediately from Proposition 5.27. □

An important application of the above analysis is that there is essentially only one infinite dimensional separable Hilbert space.

Definition 5.31. Let $(V, \|\cdot\|)$ and $(V', \|\cdot\|')$ be two normed vector spaces (over the same field). A map $T: V \to V'$ is called an *isometric isomorphism* if it is a vector space isomorphism and if, for all $x \in V$, $\|Tx\|' = \|x\|$ (i.e. T is an isometry of the two vector spaces). If there is an isometric isomorphism between two normed vector spaces then we say that the spaces are *isometrically isomorphic*.

Just as two isomorphic vector spaces cannot be distinguished as vector spaces, so two isometrically isomorphic normed vector spaces cannot be distinguished as normed vector spaces and, in this context, we regard them as equal.

Theorem 5.32. *All infinite dimensional separable Hilbert spaces over $\mathbb{C}$ are isometrically isomorphic to ℓ^2. (Similarly, all infinite dimensional separable Hilbert spaces over $\mathbb{R}$ are isometrically isomorphic to $\ell^2(\mathbb{R})$.)*

Proof. Suppose H is an infinite dimensional separable Hilbert space. By Proposition 5.26, it contains a complete orthonormal set $\{e_n\}_{n=1}^{\infty}$. By Proposition 5.24 and Theorem 5.25, every $x \in H$ may be written in the form $\sum_{n=1}^{\infty} a_n e_n$, for some $(a_n)_{n=1}^{\infty} \in \ell^2$ and the map

$$T: H \to \ell^2 : \sum_{n=1}^{\infty} a_n e_n \mapsto (a_n)_{n=1}^{\infty}$$

is a bijection. It is easy to check that T is linear and Theorem 5.25 also gives that $\|T(x)\| = \|x\|$, i.e., T is an isometry. □

5.6 The Hilbert space $L^2([-\pi,\pi],\mathbb{R})$

We now discuss the Hilbert space $L^2([-\pi,\pi],\mathbb{R})$ with the inner product

$$\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f g \, dx.$$

(The factor $1/\pi$ is just chosen to make the functions $\cos(nx)$ and $\sin(nx)$ orthonormal.) and the relation between the above discussion and the theory of Fourier series. We recall that for any integrable function $f: [-\pi,\pi] \to \mathbb{R} \cup \pm\infty$, we may write down its Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right),$$

where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$ and, for $n \geq 1$,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx, \ n \geq 1.$$

The integrals only depend on the equivalence class of f and so the Fourier series is well defined on $L^2([-\pi,\pi],\mathbb{R}) \subset L^1([-\pi,\pi],\mathbb{R})$.

Remark 5.33. One usually defines Fourier series for functions satisfying $f(-\pi) = f(\pi)$, which extend to 2π-periodic functions, but since elements of $L^2([-\pi,\pi],\mathbb{R})$ are only defined up to equality almost everywhere the restriction becomes irrelevant. However, the condition is required for continuous functions below.

Define

$$\phi_n(x) = \begin{cases} \sin(-nx) \text{ if } n < 0 \\ \frac{1}{\sqrt{2}} \text{ if } n = 0 \\ \cos(nx) \text{ if } n > 0. \end{cases}$$

It is a straightforward calculation that $\{\phi_n\}_{n\in\mathbb{Z}}$ is an orthonormal subset of $L^2([-\pi,\pi],\mathbb{R})$. Moreover, $\langle f, 1/\sqrt{2}\rangle = a_0/\sqrt{2}$ and, for $n \geq 1$,

$$\langle f, \phi_n\rangle = a_n, \qquad \langle f, \phi_{-n}\rangle = b_n.$$

Hence the Fourier series for f has the succinct expression

$$\sum_{n=-\infty}^{\infty} \langle f, \phi_n\rangle\phi_n.$$

Theorem 5.34. *The set $\{\phi_n\}_{n\in\mathbb{Z}}$ is an orthonormal basis for $L^2([-\pi,\pi],\mathbb{R})$ and, for $f \in L^2([-\pi,\pi],\mathbb{R})$,*

$$\lim_{n\to+\infty}\left\| f - \sum_{k=-n}^{n}\langle f, \phi_k\rangle\phi_k\right\|_2 = 0,$$

i.e. a function is equal to its Fourier series in $L^2([-\pi,\pi],\mathbb{R})$.

Proof. The theorem will follow from Lemma 5.30 and the Riesz-Fischer Theorem once we show that $\{\phi_n\}_{n\in\mathbb{Z}}$ is complete. Suppose that $f \in L^2([-\pi,\pi],\mathbb{R})$ is such that $\langle f, \phi_n\rangle = 0$ for all $n \in \mathbb{Z}$. Recall that $L^2([-\pi,\pi],\mathbb{R})$ is the $\|\cdot\|_2$-completion of $C([-\pi,\pi],\mathbb{R})$. This means that given $\epsilon > 0$ there exists $g \in C([-\pi,\pi],\mathbb{R})$ such that $\|f - g\|_2 < \epsilon/2$. With a little extra work, one can choose g so that $g(-\pi) = g(\pi)$. By Example 3.9, we can find $n \geq 0$ and scalars $c_{-n}, \ldots, c_n$ such that

$$\left\| g - \sum_{k=-n}^{n} c_k\phi_k\right\|_2 \leq \sqrt{2}\left\| g - \sum_{k=-n}^{n} c_k\phi_k\right\|_\infty < \epsilon/2.$$

Hence,

$$\left\| f - \sum_{k=-n}^{n} c_k\phi_k\right\|_2 \leq \|f - g\|_2 + \left\| g - \sum_{k=-n}^{n} c_k\phi_k\right\|_2 < \epsilon.$$

However, by Theorem 5.21, $\sum_{k=-n}^{n} \langle f, \phi_k \rangle \phi_k = 0$ is the closest point to f in $\overline{\text{span}(\{\phi_k\}_{k=-n}^{n}}$, so that we also have

$$\|f\|_2 = \left\| f - \sum_{k=-n}^{n} \langle f, \phi_k \rangle \phi_k \right\|_2 < \epsilon.$$

Since $\epsilon > 0$ is arbitrary, this shows that $\|f\|_2 = 0$ and hence that $f = 0$ in $L^2([-\pi, \pi], \mathbb{R})$. Therefore $\{\phi_n\}_{n \in \mathbb{Z}}$ is complete. □

The existence of a basis shows that $L^2([-\pi, \pi], \mathbb{R})$ is separable (see exercises), though this could also be proved directly in a variety of ways. As we have seen above, this means that $L^2([-\pi, \pi], \mathbb{R})$ is isometrically isomorphic to $\ell^2(\mathbb{R})$. We can actually write down the isometric isomorphism explicitly. For notational convenience, we relabel the indexing of the elements of $\ell^2(\mathbb{R})$ and write

$$\ell^2(\mathbb{R}) = \left\{ (a_n)_{n \in \mathbb{Z}} : \sum_{n \in \mathbb{Z}} |a_n|^2 < +\infty, \ a_n \in \mathbb{R} \right\}.$$

Then the map $T : H = L^2([-\pi, \pi], \mathbb{R}) \to \ell^2(\mathbb{R})$ used in the proof of Theorem 5.32 is just the map that sends a function to its Fourier coefficients:

$$T(f) = (\langle f, \phi_n \rangle)_{n \in \mathbb{Z}}.$$

The fact that T is an isometry can be expressed as

$$\|f\|_2 = \frac{1}{\pi} \int_{-\pi}^{\pi} |f|^2 \, dx = \sum_{n \in \mathbb{Z}} |\langle f, \phi_n \rangle|^2,$$

which is the standard Parseval identity for Fourier series.

6 The Dual Space

6.1 Continuous and bounded functionals

Definition 6.1. Let V be a (normed) vector space over $\mathbb{R}$ (or $\mathbb{C}$). A *linear functional* on V is a map $f : V \to \mathbb{R}$ (or $\mathbb{C}$) such that

$$f(\lambda x + \mu y) = \lambda f(x) + \mu f(y),$$

for all $x, y \in V$, $\lambda, \mu \in \mathbb{R}$ (or $\mathbb{C}$).

Example 6.2. Let $V = \mathbb{R}^n$. Then every linear functional $f : \mathbb{R}^n \to \mathbb{R}$ has the form

$$f(x) = \sum_{i=1}^{n} a_i x_i = \langle a, x \rangle,$$

where $x = (x_1, \dots, x_n)$ and $a = (a_1, \dots, a_n) \in \mathbb{R}^n$.

Example 6.3. Let $V = C([0,1], \mathbb{R})$. Then an example of a linear functional $f : C([0,1], \mathbb{R}) \to \mathbb{R}$ is given by

$$f(\phi) = \int_0^1 \phi(x)dx.$$

Definition 6.4. The *dual space* of V, denoted by V^*, is the set of all *continuous* linear functionals on V, i.e., those for which $\lim_{n\to+\infty} \|x_n - x\| = 0$ implies $\lim_{n\to+\infty} f(x_n) = f(x)$. (We shall see later that V^* is itself a vector space.)

Definition 6.5. A linear functional f is called *bounded* if there exists $M \geq 0$ such that

$$|f(x)| \leq M\|x\|, \quad \text{for all } x \in V.$$

Proposition 6.6. *Let $f : V \to \mathbb{R}$ (or $\mathbb{C}$) be a linear functional on a normed vector space V. Then f is continuous if and only if f is bounded.*

Proof. ($\implies$) Suppose that f is continuous. Assume (for a contradiction) that there is no $M \geq 0$ for which $|f(x)| \leq M\|x\|$, for all $x \in V$. Then we can choose a sequence $x_n \in V$, $n \geq 1$, such that $|f(x_n)| > n\|x_n\|$, so that

$$\left| f\left(\frac{1}{n}\frac{x_n}{\|x_n\|}\right) \right| = \frac{|f(x_n)|}{n\|x_n\|} > 1.$$

On the other hand,

$$\left\| \frac{1}{n}\frac{x_n}{\|x_n\|} \right\| \to 0, \text{ as } n \to +\infty,$$

so, by continuity at 0,

$$f\left(\frac{1}{n}\frac{x_n}{\|x_n\|}\right) \to f(0) = 0, \text{ as } n \to +\infty,$$

giving the required contradiction.
($\Longleftarrow$) Suppose that f is bounded. Given $x \in V$ and $\epsilon > 0$, we need to show that there exists $\delta > 0$ such that $\|x - y\| < \delta$ implies that $|f(x) - f(y)| < \epsilon$. If $M = 0$ then $|f(x) - f(y)| = |f(x - y)| = 0$, so any δ will do. If $M > 0$, choose $\delta = \epsilon/M$. Then, if $\|x - y\| < \delta$,

$$|f(x) - f(y)| = |f(x - y)| \leq M\|x - y\| < M\frac{\epsilon}{M} = \epsilon,$$

as required. □

Definition 6.7. Let V be a normed vector space (with norm $\|\cdot\|$). Then we define a norm on V^* by

$$\|f\| = \sup_{\|x\|=1} |f(x)|.$$

By Proposition 6.6, this is finite and an equivalent definition is

$$\|f\| = \sup_{\|x\|\neq 0} \frac{|f(x)|}{\|x\|}.$$

(N.B. it still has to be proved that this is a norm – we do this below.)

An immediate consequence of the definition is the following estimate. We shall use it frequently.

Corollary 6.8. *For all $x \in V$,*

$$|f(x)| \leq \|f\|\,\|x\|.$$

Example 6.9. $V = C([0,1], \mathbb{R})$ (with $\|\cdot\|_\infty$),

$$f(\phi) = \int_0^1 \phi(x)dx.$$

Then

$$|f(\phi)| \leq \|\phi\|_\infty \quad \text{so} \quad \|f\| \leq 1.$$

Putting $\phi(x) = 1\ \forall x \in [0,1]$ gives

$$\|f\| \geq |f(1)| = 1.$$

Thus $\|f\| = 1$.

Example 6.10. $V = \ell^1$ (with $\|\cdot\|_1$),

$$f((x_i)_{i=1}^\infty) = x_1.$$

Then

$$|f((x_i)_{i=1}^\infty)| = |x_1| \leq \sum_{i=1}^\infty |x_i| = \|(x_i)_{i=1}6\infty)\|_1 \quad \text{so} \quad \|f\| \leq 1.$$

Putting $(x_i)_{i=1}^\infty = (1,0,0,\ldots)$ gives

$$\|f\| \geq |f((1,0,0,\ldots))| = 1.$$

Thus $\|f\| = 1$.

Proposition 6.11. *If $(V, \|\cdot\|)$ is a normed vector space then $(V^*, \|\cdot\|)$ is a Banach space.*

Proof. V^* is a vector space: Suppose that $f, g \in V^*$ and that λ is a scalar. Then

$$(\lambda f)(x) = \lambda f(x)$$

and

$$(f+g)(x) = f(x) + g(x)$$

and these are clearly continuous.
$\|\cdot\|$ is a norm:

(1) Clearly $\|f\| \geq 0$ and

$$\|f\| = 0 \quad \iff \quad \sup_{\|x\|=1} |f(x)| = 0 \quad \iff \quad |f(x)| = 0, \text{ for all } x \text{ with } \|x\| = 1$$

and the latter identity is equivalent to $f = 0$.

(2)

$$\|\lambda f\| = \sup_{\|x\|=1} |\lambda||f(x)| = |\lambda| \sup_{\|x\|=1} |f(x)| = |\lambda|\|f\|.$$

(3) For $\|x\| = 1$,

$$\begin{aligned}|f(x) + g(x)| &\leq |f(x)| + |g(x)| \\ &\leq \sup_{\|x\|=1} |f(x)| + \sup_{\|x\|=1} |g(x)| = \|f\| + \|g\|.\end{aligned}$$

Taking the supremum, we get

$$\|f + g\| = \sup_{\|x\|=1} |f(x) + g(x)| \leq \|f\| + \|g\|.$$

V^* is a Banach space: Suppose that $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence in V^*. Fixing $x \in V$,

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\|\|x\|,$$

so $\{f_n(x)\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$ (or $\mathbb{C}$) and we may write $f(x) = \lim_{n\to+\infty} f_n(x)$. We need to show that this f is an element of V^*.

First we check that f is linear:

$$f(\lambda x + \mu y) = \lim_{n\to+\infty} f_n(\lambda x + \mu y) = \lim_{n\to+\infty} (\lambda f_n(x) + \mu f_n(y)) = \lambda f(x) + \mu f(y).$$

Next, we check that f is bounded (remember this is equivalent to continuous). Since $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence, we may choose $N \geq 1$ so that $n, m \geq N$ implies that $\|f_n - f_m\| \leq 1$. We have, for $\|x\| = 1$ and $n \geq N$,

$$\begin{aligned}|f(x)| &\leq |f(x) - f_N(x)| + |f_N(x)| \\ &= \lim_{n\to+\infty} |f_n(x) - f_N(x)| + |f_N(x)| \\ &\leq \limsup_{n\to+\infty} \|f_n - f_N\| + \|f_N\| \leq 1 + \|f_N\|,\end{aligned}$$

so f is bounded, as required.

To finish, we check that f_n converges to f in the norm $\|\cdot\|$ on V^*. Since f_n is a Cauchy sequence in V^*, given $\epsilon > 0$, there exists $N \geq 1$ such that for $n, m \geq N$ we have that

$$\|f_n - f_m\| < \epsilon.$$

Then, for any $x \in V$ with $\|x\| = 1$, we have, for $n, m \geq N$,

$$|f_n(x) - f_m(x)| < \epsilon.$$

Letting $m \to \infty$, for all $x \in V$ with $\|x\| = 1$ and $n \geq N$, we have

$$|f_n(x) - f)x)| \leq \epsilon.$$

Hence, for $n \geq N$,

$$\|f_n - f\| = \sup_{\|x\|=1} |f_n(x) - f(x)| \leq \epsilon.$$

Thus $\lim_{n\to\infty} \|f_n - f\| = 0$. $\square$

Remark 6.12. Note that we did not need to assume that V is a Banach space.

6.2 Linear Functionals on Hilbert Spaces

Let H be a Hilbert space (with inner product $\langle \cdot, \cdot \rangle$) and choose $y \in H$. Then $f_y(x) = \langle x, y \rangle$ is a bounded (continuous) linear functional. We have

$$\|f_y\| = \sup_{x \neq 0} \frac{|\langle x, y \rangle|}{\|x\|}.$$

By the Cauchy-Schwarz inequality,

$$\frac{|\langle x, y \rangle|}{\|x\|} \leq \|y\|,$$

with equality for (in particular) $x = y$. Therefore, $\|f_y\| = \|y\|$.

We shall now show that every element of H^* can be written in the above form. More formally, we can think of the above construction as a map

$$H \to H^* : x \mapsto f_x.$$

Definition 6.13. A linear map $T : V \to V'$ between two normed vector spaces $(V, \|\cdot\|)$ and $(V', \|\cdot\|')$ is called an *isometric isomorphism* if it is a bijection and satisfies $\|Tx\|' = \|x\|$, for all $x \in V$. We then write $V = V'$.

Theorem 6.14 (Riesz Representation Theorem for Hilbert spaces = Riesz-Frechet Theorem)**.** *Every element of H^* has the form f_x, fir some $x \in H$. Furthermore, the map*

$$H \to H^* : x \mapsto f_x$$

is an isometric isomorphism if H is over $\mathbb{R}$ and a conjugate linear (i.e. $\lambda x + \mu y \mapsto \overline{\lambda} f_x + \overline{\mu} f_y$) isometric bijection if H is over $\mathbb{C}$.

Proof. That the map is linear or conjugate linear is clear and we have already seen that $\|f_x\| = \|x\|$. Furthermore, if $x \neq y$ then $\|f_x - f_y\| = \|f_{x-y}\| = \|x - y\| > 0$, so $x \mapsto f_x$ is injective. Thus it only remains to show that $x \mapsto f_x$ is surjective.

Choose $f \in H^*$, $f \neq 0$, and let

$$L = \ker f = \{y \in H : f(y) = 0\}.$$

Since $f \neq 0$, we have $L \neq H$, so $L^\perp \neq \{0\}$. Thus we can choose $x_0 \in L^\perp$ (so, in particular, $f(x_0) \neq 0$).

Given $y \in H$, let

$$w = y - \frac{f(y)}{f(x_0)} x_0.$$

Then

$$f(w) = f\left(y - \frac{f(y)}{f(x_0)} x_0\right) = f(y) - \frac{f(y)}{f(x_0)} f(x_0) = 0,$$

so $w \in L$. However, since $x_0 \in L^\perp$, this gives

$$0 = \langle w, x_0 \rangle = \langle y, x_0 \rangle - \langle x_0, x_0 \rangle \frac{f(y)}{f(x_0)},$$

i.e.,

$$\langle y, x_0 \rangle = \langle x_0, x_0 \rangle \frac{f(y)}{f(x_0)}.$$

Rearranging, we get

$$f(y) = \left\langle y, \frac{\overline{f(x_0)}}{\|x_0\|^2} x_0 \right\rangle = f_v(y),$$

where

$$v = \frac{\overline{f(x_0)}}{\|x_0\|^2} x_0.$$

So, $f = f_v$ and we see that $x \mapsto f_x$ is surjective. □

Remark 6.15. If H is defined over $\mathbb{C}$ and is separable then we have seen that it is isometrically isomorphic to ℓ^2. We can define an isometric isomorphism between ℓ^2 and $(\ell^2)^*$ by mapping $(x_n)_{n=1}^\infty$ to $f_{(\overline{x_n})_{n=1}^\infty}$, where

$$f_{(\overline{x_n})_{n=1}^\infty}((y_n)_{n=1}^\infty = \sum_{n=1}^\infty y_n x_n.$$

6.3 Linear functionals on ℓ^p

Suppose that $1 < p, q < +\infty$ satisfy

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Put $a = (a_1, a_2, \ldots) \in \ell^p$ and $b = (b_1, b_2, \ldots) \in \ell^q$. For each fixed $b \in \ell^q$, define

$$f_b(a) = \sum_{i=1}^{\infty} a_i b_i.$$

Theorem 6.16. *The linear map*

$$T : \ell^q \to (\ell^p)^* : b \mapsto f_b,$$

is an isometric isomorphism, so $(\ell^p)^* = \ell^q$.

In the case $p = q = 2$, we discussed this result above (since ℓ^2 is a Hilbert space).

Proof. First note that

$$f_b(a) = \sum_{i=0}^{\infty} a_i b_i$$

is finite since, by Hölder's inequality,

$$|f_b(a)| \leq \sum_{i=1}^{\infty} |a_i b_i| \leq \left(\sum_{i=1}^{\infty} |a_i|^p \right)^{1/p} \left(\sum_{i=1}^{\infty} |b_i|^q \right)^{1/q} = \|a\|_p \|b\|_q. \qquad (*)$$

Clearly f_b is linear and $(*)$ shows that f is bounded and, furthermore, that $\|f_b\| \leq \|b\|_q$.

To show equality, we shall make a particular choice for a, depending on b. If $b_i = |b_i|e^{i\theta}$ then we choose $a_i = |b_i|^{q/p} e^{-i\theta}$. Then $a_i b_i = |a_i b_i|$ and $|a_i|^p / |b_i|^q = 1$. Thus

$$\|f_b\| \|a\|_p \geq |f_b(a)| = \sum_{i=1}^{\infty} a_i b_i = \|a\|_p \|b\|_q$$

(using the equality statement in Hölder's inequality). Therefore $\|f_b\| = \|b\|_q$.

It remains to show that $T : \ell^q \to (\ell^p)^*$ is an isomorphism. It is clear that T is linear. Also, if $b \neq b'$ then

$$\|T(b) - T(b')\| = \|T(b - b')\| = \|b - b'\| > 0,$$

so $T(b) \neq T(b')$ and T is injective. (Isometry always implies injectivity by this argument.) Finally, we show that T is surjective. Given $f \in (\ell^p)^*$, write $f_n = f(e_n)$, where

$$e_n = (\underbrace{0, \ldots, 0}_{n-1}, 1, 0, \ldots).$$

It will be enough to show that $(f_n)_{n=0}^{\infty} \in \ell^q$, since then $T((f_n)_{n=0}^{\infty}) = f$.
Claim: $x = (f_n)_{n=0}^{\infty} \in \ell^q$. Assume for a contradiction that

$$\sum_{n=0}^{\infty} |f_n|^q = +\infty.$$

Let $x_N = (f_0, f_1, \ldots, f_N, 0, \ldots)$, then $\|x_N\|_q \to +\infty$, as $N \to +\infty$.

Fix N and let

$$y_N = \left(e^{-i\theta_0} \left(\frac{|f_0|^q}{\|x_N\|_q^q} \right)^{1/p}, e^{-i\theta_1} \left(\frac{|f_1|^q}{\|x_N\|_q^q} \right)^{1/p}, \ldots, e^{-i\theta_N} \left(\frac{|f_N|^q}{\|x_N\|_q^q} \right)^{1/p}, 0, \ldots \right),$$

where

$$f_n = e^{i\theta_n}|f_n|.$$

Then

$$\|y_N\|_p = \left(\sum_{n=0}^{N} \frac{|f_n|^q}{\|x_N\|_q^q} \right)^{1/p} = \left(\frac{\|x_N\|_q^q}{\|x_N\|_q^q} \right)^{1/p} = 1$$

and, in particular, $|(y_N)_n|^p = c|f_n|^q$, where $c = \|x_N\|_q^{-q}$.

By Hölder's inequality, we have

$$\begin{aligned} f(y_N) &= \sum_{n=0}^{N} (y_N)_n f_n \\ &= \sum_{n=0}^{N} (y_N)_n (x_N)_n \\ &= \|x_N\|_q \|y_N\|_p = \|x_N\|_q \end{aligned}$$

(since $\|y_N\|_p = 1$). Thus $\|x_N\|_q = |f(y_N)| \leq \|f\|$, which contradicts $\|x_N\|_q \to +\infty$, as $N \to +\infty$.

Thus we conclude that $x = (f_n)_{n=0}^{\infty} \in \ell^q$, and the proof is complete. $\square$

Now consider $\ell^1 = \left\{ a = (a_i)_{i=1}^{\infty} : \sum_{i=1}^{\infty} |a_i| < +\infty,\ a_i \in \mathbb{C} \right\}$. Fix $b = (b_i)_{i=1}^{\infty} \in \ell^\infty$ and define

$$f_b(a) = \sum_{i=0}^{\infty} a_i b_i.$$

Theorem 6.17. *The linear map*

$$T : \ell^\infty \to (\ell^1)^* : b \mapsto f_b$$

is an isometric isomorphism, so $(\ell^1)^* = \ell^\infty$.

Proof. The sum

$$f_b(a) = \sum_{i=0}^{\infty} a_i b_i$$

is finite since

$$\left| \sum_{i=1}^{\infty} a_i b_i \right| \leq \sum_{i=1}^{\infty} |a_i b_i| \leq \left(\sum_{i=1}^{\infty} |a_i| \right) \left(\sup_{i \geq 1} |b_i| \right) = \|a\|_1 \|b\|_\infty.$$

Clearly f_b is linear and the above inequality also shows that $\|f_b\| \leq \|b\|_\infty$.

To show equality, we shall make a particular choices for a, depending on b. Given $\epsilon > 0$, choose $j \geq 1$ so that $|b_j| > \|b\|_\infty - \epsilon$. If $b_j = |b_j|e^{i\theta}$ then we choose

$$a_i = \begin{cases} e^{-i\theta} & \text{if } i = j \\ 0 & \text{of } i \neq j. \end{cases}$$

Then $a_j b_j = |a_j b_j| = |b_j|$ and $\|a\|_1 = |a_j| = 1$. Thus

$$\|f_b\| = \|f_b\|\|a\|_1 \geq |f_b(a)| = \sum_{i=1}^{\infty} a_i b_i = a_j b_j > \|b\|_\infty - \epsilon.$$

Since $\epsilon > 0$ is arbitrary, this gives us $\|f_b\| \geq \|b\|_\infty$, as required.

As in the preceding proof, T is linear and, since it is an isometry, it is injective. To show that T is surjective, choose $f \in (\ell^1)^*$ and consider $f_n = f(e_n)$, $n \geq 0$. We have

$$|f_n| = |f(e_n)| \leq \|f\|\|e_n\|_1 = \|f\|,$$

so we see that $(f_n)_{n=0}^\infty \in \ell^\infty$. Clearly, $T((f_n)_{n=0}^\infty)$; thus T is surjective. This completes the proof. □

Remark 6.18. The converse to this result is not true: $(\ell^\infty)^* \neq \ell^1$. In fact, $(\ell^\infty)^*$ is equal to a space called $\mathrm{ba}(\mathbb{N})$, which is strictly larger than ℓ^1. This space has the following definition. Let $\mathcal{P}(\mathbb{N})$ denote the set of all subsets of $\mathbb{N}$. We say that a function $\mu : \mathcal{P}(\mathbb{N}) \to \mathbb{R}$ is bounded if there exists $M \geq 0$ such that $|\mu(E)| \leq M$ for all $E \in \mathcal{P}(\mathbb{N})$. We say that $\mu : \mathcal{P}(\mathbb{N}) \to \mathbb{R}$ is finitely additive if, for all $E, F \in \mathcal{P}(\mathbb{N})$ with $E \cap F = \varnothing$, we have $\mu(E \cup F) = \mu(E) + \mu(F)$. Then $\mathrm{ba}(\mathbb{N})$, is defined to be the set of all bounded finitely additive functions $\mu : \mathcal{P}(\mathbb{N}) \to \mathbb{R}$. We shall not discuss the proof that $(\ell^\infty)^* = \mathrm{ba}(\mathbb{N})$, which is rather complicated.

Definition 6.19. We define a strictly smaller space $c_0 \subset \ell^\infty$ by

$$c_0 = \{(b_i)_{i=0}^\infty \in \ell^\infty : b_i \to 0, \quad \text{as } i \to +\infty\}.$$

Proposition 6.20. $(c_0)^* = \ell^1$.

Proof. See exercises. □

6.4 The dual space of the continuous functions

Let X be a compact metric space and let $\mathcal{M}(X)$ denote the set of all finite Borel measures on X (i.e. measures for which sets in the Borel σ-algebra are measurable). For $\mu \in \mathcal{M}(X)$, the map $w_\mu : C(X, \mathbb{R}) \to \mathbb{R}$ defined by

$$w_\mu(f) = \int f \, d\mu$$

is a linear functional. (Note that here f is an element of the original vector space, not the dual space.) Furthermore, the inequality

$$\left| \int_X f \, d\mu \right| \leq \mu(X) \|f\|_\infty$$

(which holds for all measures) shows that w_μ is bounded, so $w_\mu \in C(X, \mathbb{R})^*$. In fact, every *positive* bounded linear functional is of this form (where w positive means that $f \geq 0 \implies w(f) \geq 0$). To obtain all bounded linear functionals, we take differences of the w_μ. This is summarised in the following theorem.

Theorem 6.21 (Riesz Representation Theorem for continuous functions)**.** *Let X be a compact metric space. Then*

$$C(X, \mathbb{R})^* = \{w_\mu - w_\nu : \mu, \nu \in \mathcal{M}(X)\},$$

where w_μ is the functional defined above.

6.5 The weak* topology

We have defined a norm on V^* and this norm defines a topology. We call this the *strong topology* (or *norm topology*) on V^*. However, there is another, very different, topology that can be defined on V^*. This is called the *weak* topology*.

Definition 6.22. The weak* topology on V^* is the smallest topology such that the maps $\delta_x : V^* \to \mathbb{R}$ (or $\mathbb{C}$) defined by $\delta_x(f) = f(x)$ are continuous for all $x \in V$.

A topology on a set X is exactly determined by specifying when, for $x \in X$ and sequences $x_n \in X$, $x_n \to x$, as $n \to +\infty$. (As an exercise, convince yourself that this is true.) We will use this criterion to give an alternative definition of the topology.

Definition 6.23. Let f_n, $n \geq 1$ be a sequence in V^*. We say that $f_n \to f \in V^*$, as $n \to +\infty$, in the weak* topology if, for all $x \in V$,

$$\lim_{n \to +\infty} f_n(x) = f(x).$$

Lemma 6.24. *If f_n converges to f in the strong topology then it also converges to f is the weak* topology.*

Proof. The result follows from the inequality

$$|f_n(x) - f(x)| = |(f_n - f)(x)| \leq \|f_n - f\| \|x\|.$$

□

However, except in finite dimensions (in which case the two topologies agree), the converse is not true.

That the two topologies are different is spectacularly illustrated by the following theorem. Let $B = \{f \in V^* : \|f\| \leq 1\}$. Recall that, for an infinite dimensional normed vector space, the closed unit ball is not compact (Theorem 2.31). Hence, if V^* is infinite dimensional then then B is not compact in the strong topology. However, B is compact in the weak* topology.

Theorem 6.25 (Banach-Alaoglu Theorem)**.** *The closed unit ball B is compact in the weak* topology on V^*.*

Proof. Note that
$$B \subset Y = \prod_{x \in V} \{\lambda \in \mathbb{C} : |\lambda| \leq \|x\|\},$$
because every element of B is an assignment of a number $f(x) \in \mathbb{C}$ to every $x \in V$ satisfying $|f(x)| \leq \|x\|$. Y is a product of compact sets and so, by Tychonoff's Theorem, is compact in the product topology. With some thought, one can see that the restriction of this topology to B is exactly the weak* topology restricted to B. Hence, the theorem will follow if we can show that B is closed in Y (since a closed subset of a compact set is compact.

Suppose that $f \in Y$ lies in the closure of B, Then, given $x, y \in V$, $a, b \in \mathbb{C}$, for all sufficiently small $\delta > 0$, the sets $\{g \in Y : |(g - f)(x)| < \delta\}$, $\{g \in Y : |(g - f)(y)| < \delta\}$ and $\{g \in Y : |(g - f)(ax + by)| < \delta\}$ are open in Y and contain f. Hence they have non-empty intersection with B. Let F be an element of this intersection. Then
$$\begin{aligned} &|af(x) + bf(y) - f(ax + by)| \\ &\leq |a||f(x) - F(x)| + |b||f(y) - F(y)| + |f(ax + by) - F(ax + by)| \\ &+ |aF(x) + bF(y) - F(ax + by)| \\ &\leq (|a| + |b|)\delta. \end{aligned}$$
Since $\delta > 0$ can be taken arbitrarily small, this shows that f is linear.

Now, given $\epsilon > 0$, let N be a neighbourhood of $0 \in V$ such that $x - y \in N$ implies that $|F(x) - F(y)| < \epsilon$. Then
$$|f(x) - f(y)| \leq |f(x) - F(x)| + |F(x) - F(y)| + |F(y) - f(y)| < 2\delta + \epsilon.$$
This shows that f is continuous. Thus $f \in B$ and B is closed. $\square$

7 Linear Operators and Spectra

In the preceding section, we looked at linear maps valued in the field of scalars $\mathbb{R}$ or $\mathbb{C}$. Now we will generalize this to linear maps between two normed vector spaces. We begin by recalling the definition of a linear map.

Definition 7.1. Let V and V' be normed vector spaces. A *linear operator* is a map $T : V \to V'$ such that

$$T(\lambda x + \mu y) = \lambda T(x) + \mu T(y),$$

for all $x, y \in V$, $\lambda, \mu \in \mathbb{R}$ (or $\mathbb{C}$).

The following is analogous to Proposition 6.6 for linear functionals. To make notation less cumbersome, we shall denote both the norm on V and the norm on V' by $\|\cdot\|$.

We shall be interested in linear operators $T : V \to V'$ which are continuous, i.e., those for which $\lim_{n\to+\infty} \|x_n - x\| = 0$ implies $\lim_{n\to+\infty} \|T(x_n) - T(x)\| = 0$.

Definition 7.2. We say that $T : V \to V'$ is bounded if there exists $M \geq 0$ such that

$$\|T(x)\| \leq M\|x\|, \quad \text{for all } x \in V.$$

Theorem 7.3. *Let $T : V \to V'$ be a linear operator between the normed vector spaces V and V'. Then T is continuous if and only if T is bounded.*

Proof. Exercise. (It is essentially the same as the proof of the corresponding result for linear functionals, Proposition 6.6.) □

Definition 7.4. If $T : V \to V'$ is a bounded linear operator then we define its norm $\|T\|$ by

$$\|T\| = \sup_{\|x\|=1} |T(x)|.$$

By Theorem 7.3, this is finite and an equivalent definition is

$$\|T\| = \sup_{\|x\|\neq 0} \frac{\|T(x)\|}{\|x\|}.$$

(N.B. it still has to be proved that this is a norm.)

An immediate consequence of the definition is the following estimate (cf. Corollary 6.8).

Corollary 7.5. *For all $x \in V$,*

$$\|T(x)\| \leq \|T\| \, \|x\|.$$

Definition 7.6. We define $B(V, V')$ to be the set of all bounded linear operators $T : V \to V'$. (If V is over $\mathbb{R}, \mathbb{C}$ and $V' = \mathbb{R}, \mathbb{C}$ then $B(V, V') = V^*$.)

Proposition 7.7. *If V' is a Banach space then so is $B(V, V')$. (We do not need to assume that V is a Banach space.*

Proof. Exercise. (It is essentially the same as the proof of Proposition 6.11 for linear functionals.) □

Now we will consider the case where $V = V'$. We then use the simpler notation

$$B(V) = B(V, V).$$

If $T, T' \in B(V)$ we write TT' for their composition: $(TT')(x) = T(T'(x))$. This is also a linear operator.

Proposition 7.8. *If $T, T' \in B(V)$ then $\|TT'\| \leq \|T\| \, \|T'\|$. (In particular, $TT' \in B(V)$.)*

Proof. Applying Corollary 7.5 twice, we have

$$\|TT'(x)\| = \|T(T'(x))\| \leq \|T\| \, \|T'(x)\| \leq \|T\| \, \|T'\| \, \|x\|.$$

Thus TT' is bounded (so $TT' \in B(V, V)$) and $\|TT'\| \leq \|T\| \, \|T'\|$. □

Example 7.9. If $V = \mathbb{R}^n$ then $T \in B(V)$ is given by an $n \times n$ real matrix.

Similarly, If $V = \mathbb{C}^n$ then $T \in B(V)$ is given by an $n \times n$ complex matrix.

Example 7.10. Let $V = \ell^p$, $1 \leq p < \infty$. A particular $T \in B(V)$ is the *shift* $T : \ell^p \to \ell^p$ defined by

$$T(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots).$$

Then ($x = (x_1, x_2, x_3, \ldots)$)

$$\|T(x)\| = \left(\sum_{i=2}^{\infty} |x_i|^p \right)^{1/p} \leq \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p} = \|x\|,$$

so $\|T\| \leq 1$.

Now put $x = (0, x_2, x_3, \ldots)$, so $T(x) = (x_2, x_3, x_4, \ldots)$. For this x, it is clear that $\|T(x)\| = \|x\|$, so we have

$$\|T\| \|x\| \geq \|T(x)\| = \|x\|,$$

giving $\|T\| \geq 1$. Therefore $\|T\| = 1$.

Definition 7.11. We say that $T \in B(V, V')$ is an an isometry if $\|T(x)\| = \|x\|$, for all $x \in V$.

We have now got back to a definition we made in Chapter 3: $T : V \to V'$ is an isometric isomorphism if T is a linear isometry which is also a bijection.

7.1 Hilbert Spaces and Adjoint Operators

Let H be a Hilbert space (with inner product $\langle \cdot, \cdot \rangle$) and let $T : H \to H$ be a bounded linear operator. We want to define a bounded linear operator $T^* : H \to H$, called the *adjoint* of T, such that

$$\langle Tx, y \rangle = \langle x, T^* y \rangle, \quad \text{for all } x, y \in H.$$

As we shall see, this generalizes the notion of transpose for real matrices or Hermitian transpose for complex matrices.

Let us see that we can do this. Fix $y \in H$ and consider the linear map

$$H \to \mathbb{C} : x \mapsto \langle Tx, y \rangle.$$

Since, using the Cauchy-Schwarz inequality,

$$|\langle Tx, y \rangle| \leq \|Tx\| \|y\| \leq \|T\| \|x\| \|y\|,$$

this linear map is a bounded linear functional. Thus, by the Riesz Representation Theorem, there is a unique $y' \in H$ such that

$$\langle Tx, y \rangle = \langle x, y' \rangle$$

and we shall define $T^* : H \to H$ by

$$T^*(y) = y'.$$

Before we show that T^* is a bounded linear operator, let's give some examples.

Example 7.12. Let $H = \mathbb{R}^n$. A linear operator in $B(\mathbb{R}^n)$ is given by an $n \times n$ real matrix $A = (a_{ij})$ with $(Ax)_i = \sum_{j=1}^n a_{ij} x_j$. Now

$$\langle Ax, y \rangle = \sum_{i=1}^n (Ax)_i y_i = \sum_{i,j=1}^n a_{ij} x_j y_i = \sum_{j=1}^n \left(\sum_{i=1}^n a_{ij} y_i \right) x_j = \sum_{j=1}^n (A^t y)_j x_j = \langle x, A^t y \rangle.$$

Thus the adjoint of A is A^t (the transpose of A).

Example 7.13. Let $H = \mathbb{C}^n$. A linear operator in $B(\mathbb{C}^n)$ is given by an $n \times n$ complex matrix $A = (a_{ij})$ with $(Ax)_i = \sum_{j=1}^n a_{ij} x_j$. Now

$$\begin{aligned} \langle Ax, y \rangle &= \sum_{i=1}^n (Ax)_i \overline{y_i} = \sum_{i,j=1}^n a_{ij} x_j \overline{y_i} = \sum_{j=1}^n \left(\sum_{i=1}^n a_{ij} \overline{y_i} \right) x_j \\ &= \sum_{j=1}^n \overline{\left(\sum_{i=1}^n \overline{a_{ij}} y_i \right)} x_j = \sum_{j=1}^n \overline{(A^* y)_j} x_j = \langle x, A^* y \rangle, \end{aligned}$$

where A^* is the Hermitian transpose of A, i.e., the matrix with i, j-entry $\overline{a_{ji}}$. Thus the adjoint of A is the Hermitian transpose of A.

Example 7.14. Let $H = \ell^2$. If $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$ then

$$\langle x, y \rangle = x_1\overline{y_1} + x_2\overline{y_2} + \cdots .$$

Now let T be the shift operator we considered above:

$$T(x_1, x_2, \dots) = (x_2, x_3, \dots).$$

Then

$$\langle Tx, y \rangle = x_2\overline{y_1} + x_3\overline{y_2} + \cdots = (x_1 \times 0) + x_2\overline{y_1} + x_3\overline{y_2} + \cdots .$$

Equating this to $\langle x, T^*y \rangle$ gives

$$T^*y = (0, y_1, y_2, \dots).$$

Before we show that the adjoint T^* is indeed a bounded linear operator, we need a lemma.

Lemma 7.15. $T^{**} := (T^*)^* = T$.

Proof. By its definition,

$$\langle T^*y, x \rangle = \langle y, T^{**}x \rangle, \quad \text{for all } x, y \in H.$$

But

$$\langle T^*y, x \rangle = \overline{\langle x, T^*y \rangle} = \overline{\langle Tx, y \rangle} = \langle y, Tx \rangle.$$

Subtracting one equation from the other gives

$$0 = \langle y, Tx - T^{**}x \rangle, \quad \text{for all } x, y \in H.$$

Substituting $y = Tx - T^{**}x$ gives

$$\|Tx - T^{**}x\|^2 = \langle Tx - T^{**}x, Tx - T^{**}x \rangle = 0.$$

Thus $Tx = T^{**}x$, for all $x \in H$, i.e., $T = T^{**}$. □

Proposition 7.16. *$T^* : H \to H$ is a bounded linear operator with $\|T^*\| = \|T\|$.*

Proof. Claim. $T^* : H \to H$ is linear: For all $x, y_1, y_2 \in H$, $\lambda, \mu \in \mathbb{R}$ (or $\mathbb{C}$),

$$\begin{aligned}
\langle x, T^*(\lambda y_1 + \mu y_2) \rangle &= \langle Tx, \lambda y_1 + \mu y_2 \rangle \\
&= \overline{\lambda}\langle Tx, y_1 \rangle + \overline{\mu}\langle Tx, y_2 \rangle \\
&= \overline{\lambda}\langle x, T^*(y_1) \rangle + \overline{\mu}\langle x, T^*(y_2) \rangle \\
&= \langle x, \lambda T^*(y_1) + \mu T^*(y_2) \rangle.
\end{aligned}$$

Thus $T^*(\lambda y_1 + \mu y_2) = \lambda T^*(y_1) + \mu T^*(y_2)$.

Claim. $\|T^*\| = \|T\|$: Since $\langle Tx, y\rangle = \langle x, T^*y\rangle$, for all $x, y \in H$, by setting $x = T^*y$, we have $\langle TT^*y, y\rangle = \langle T^*y, T^*y\rangle$. Thus

$$\begin{aligned}\|T^*y\|^2 &= \langle T^*y, T^*y\rangle = \langle TT^*y, y\rangle \\ &\leq \|TT^*y\|\ \|y\| \text{ (by the Cauchy-Schwarz inequality)} \\ &\leq \|T\|\ \|T^*y\|\ \|y\|.\end{aligned}$$

In particular, $\|T^*y\| \leq \|T\|\ \|y\|$, for all $y \in H$, i.e., $\|T^*\| \leq \|T\|$. This shows that T^* is bounded.

Since $T^{**} = T$, we see that we also have $\|T\| = \|(T^*)^*\| \leq \|T^*\|$. Thus, $\|T^*\| = \|T\|$, as required. □

Proposition 7.17. *(i) $I^* = I$ (where $I : H \to H$ is the identity transformation);*

(ii) $(ST)^* = T^*S^*$;

(iii) $\|T^*T\| = \|TT^*\| = \|T\|^2$.

Proof. Property (i) follow directly from the definition. Properties (ii) and (iii) are exercises. □

Definition 7.18. We called a bounded linear operator $T : H \to H$ *self-adjoint* if $T^* = T$.

Example 7.19. An $n \times n$ real matrix A acting on $\mathbb{R}^n$ is self-adjoint if and only if $A^t = A$, i.e. A is symmetric. An $n \times n$ complex matrix A acting on $\mathbb{C}^n$ is self-adjoint if and only if A is equal to its Hermitian transpose, i.e., $\overline{a_{ji}} = a_{ij}$.

Example 7.20. Let $H = \ell^2(\mathbb{C})$ and let $(a_i)_{i=1}^\infty \in \ell^\infty(\mathbb{R})$. Define $T : \ell^2(\mathbb{C}) \to \ell^2(\mathbb{C})$ by

$$T((x_i)_{i=1}^\infty) = (a_i x_i)_{i=1}^\infty.$$

(Check that $T \in B(\ell^2(\mathbb{C}))$!) Then, for all $(x_i)_{i=1}^\infty, (y_i)_{i=1}^\infty \in \ell^2(\mathbb{C})$,

$$\begin{aligned}\langle T((x_i)_{i=1}^\infty), (y_i)_{i=1}^\infty\rangle &= \sum_{i=1}^\infty (a_i x_i)\overline{y_i} \\ &= \sum_{i=1}^\infty x_i(\overline{a_i y_i}) \\ &= \langle (x_i)_{i=1}^\infty, T((y_i)_{i=0}^\infty)\rangle.\end{aligned}$$

i.e., $T^* = T$, so T is self-adjoint. (Note that it was important for $(a_i)_{i=1}^\infty$ to heave real entries.

There is a special formula for the norm of a self-adjoint operator.

Proposition 7.21. *Let $T : H \to H$ be self-adjoint. Then*

$$\|T\| = \sup_{\|x\|=1} |\langle Tx, x\rangle|.$$

Proof. Let $M = \sup_{\|x\|=1} |\langle Tx, x\rangle|$. Since

$$|\langle Tx, x\rangle| \leq \|Tx\|\|x\| \leq \|T\|\|x\|^2,$$

we see that $M \leq \|T\|$. To prove the reverse inequality, first note that, for any $x \in H$,

$$|\langle Tx, x\rangle| \leq M\|x\|^2.$$

Now, for every $x, y \in H$, using $T^* = T$, we have

$$\begin{aligned}
\langle T(x+y), x+y\rangle &= \langle Tx + Ty, x+y\rangle \\
&= \langle Tx, x\rangle + \langle Tx, y\rangle + \langle Ty, x\rangle + \langle Ty, y\rangle \\
&= \langle Tx, x\rangle + \langle Tx, y\rangle + \langle y, T^*x\rangle + \langle Ty, y\rangle \\
&= \langle Tx, x\rangle + \langle Tx, y\rangle + \overline{\langle Tx, y\rangle} + \langle Ty, y\rangle \\
&= \langle Tx, x\rangle + 2\Re\langle Tx, y\rangle + \langle Ty, y\rangle
\end{aligned}$$

and

$$\langle T(x-y), x-y\rangle = \langle Tx, x\rangle - 2\Re\langle Tx, y\rangle + \langle Ty, y\rangle,$$

so that

$$\begin{aligned}
4\Re\langle Tx, y\rangle &= \langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle \\
&\leq M(\|x+y\|^2 + \|x-y\|^2) \\
&= 2M(\|x\|^2 + \|y\|^2).
\end{aligned}$$

We can replace x by $e^{i\theta}x$ so that $\Re\langle T(e^{i\theta}x), y\rangle = |\langle Tx, y\rangle|$. Then, for every $x, y \in H$,

$$|\langle Tx, y\rangle| \leq (M/2)(\|x\|^2 + \|y\|^2)$$

Suppose that $Tx \neq 0$. Then, taking

$$y = \frac{\|x\|}{\|Tx\|}T(x),$$

we have $\|y\| = \|x\|$ and

$$\frac{\|x\|}{\|Tx\|}|\langle Tx, Tx\rangle| \leq M\|x\|^2,$$

so that

$$\|Tx\| \leq M\|x\|.$$

This last inequality holds trivially if $Tx = 0$ and so $\|Tx\| \leq M\|x\|$ always. This shows that $\|T\| \leq M$, as required. □

7.2 Spectrum of Operators

Let V be a Banach space over $\mathbb{C}$. (Here it is important that the field is $\mathbb{C}$ (algebraically closed) not $\mathbb{R}$.)

We will say that a bounded linear operator $T \in B(V)$ is *invertible* if there exists $S \in B(V)$ such that $TS = ST = I$ (the identity transformation). If T is invertible then we denote this S by T^{-1} and call it the *inverse* of T.

Proposition 7.22. *Let V be a Banach space and let $T : V \to V$ be a continuous linear operator. If $\|I - T\| < 1$ then T is invertible.*

Proof. Since V is a Banach space, so is $B(V)$. We write $S = I - T$ and consider the sequence of operators $\{R_n\}_{n=1}^{\infty}$, where $R_n = I + S + S^2 + \cdots + S^n$ and $S^n = \underbrace{S \circ \cdots \circ S}_{\times n}$. For $n > m$, we have

$$\|R_n - R_m\| \leq \|S^{m+1} + \cdots + S^n\| \leq \|S^{m+1}\| + \cdots + \|S^n\|.$$

Since

$$\sum_{n=0}^{\infty} \|S^n\| \leq \sum_{n=0}^{\infty} \|S\|^n = \frac{1}{1 - \|S\|},$$

we see that $\{R_n\}_{n=1}^{\infty}$ is a Cauchy sequence and so it converges to some $R \in B(V, V)$.

For each $n \geq 1$, we have that

$$R_{n-1}(I - S) = (I - S)R_{n-1} = I - S^n.$$

Since $\|S^n\| \to 0$, as $n \to +\infty$, taking limits above gives $R(I - S) = (I - S)R = I$, i.e., $RT = TR = I$. Thus $R = T^{-1}$. $\square$

Recall that an $n \times n$ matrix A has eigenvalue $\lambda \in \mathbb{C}$ if any one of the following equivalent statements holds

(1) there exists $v \in \mathbb{C}^n$, $v \neq 0$, such that $Av = \lambda v$; or, equivalently,

(2) $\det(\lambda I - A) = 0$ (the characteristic equation);

(3) $\lambda I - A$ is *not* invertible.

In infinite dimensions, (2) does not make sense in general, while (1) and (3) make sense but are no longer equivalent. If (1) holds (i.e. $Tx = \lambda x$ for some non-zero $x \in V$) then we will still call λ and eigenvalue but we will focus on condition (3).

Definition 7.23. We define the *spectrum* of $T \in B(V)$ to be the set of complex numbers

$$\sigma(T) = \{\lambda \in \mathbb{C} : (\lambda I - T) : V \to V \text{ is not invertible}\}.$$

Lemma 7.24. *If λ is an eigenvalue of T then $\lambda \in \sigma(T)$.*

Proof. Suppose that $\lambda \in \mathbb{C}$ is an eigenvalue of T. Then, by definition, there exists $x \in V$, $x \neq 0$, such that

$$Tx = \lambda x$$

or, equivalently,

$$(\lambda I - T)x = 0.$$

Suppose that $\lambda \notin \sigma(T)$, so that $\lambda I - T$ is invertible. Write $S = (\lambda I - T)^{-1}$. Then

$$x = Ix = S(\lambda I - T)x = S0 = 0,$$

a contradiction, since $x \neq 0$. Therefore, $\lambda \in \sigma(T)$, as required. $\square$

An example below will show that the spectrum can contain numbers which are not eigenvalues.

Example 7.25. Let $V = \mathbb{C}^n$ and let $T : V \to V$ be a continuous linear operator. Let $e_1, \dots, e_n$ be the standard basis for $\mathbb{C}^n$, i.e., $e_k = (\underbrace{0, \dots, 0}_{\times(k-1)}, 1, 0, \dots, 0)$. Then T is represented by the $n \times n$ matrix M defined by $M(i,j) = (T(e_i))_j$, i.e., the jth entry of the vector $T(e_i)$: $T(e_i) = \sum_{j=1}^{n} M(i,j)e_j$.

The operator T is invertible $\iff$ the matrix M is non-singular $\iff$ $\det M \neq 0$.

Given $\lambda \in \mathbb{C}$, the operator $(\lambda I - T) : V \to V$ is invertible $\iff$ λ is *not* an eigenvalue for M.

If M has eigenvalues $\{\lambda_1, \dots, \lambda_n\}$ then $\sigma(T) = \{\lambda_1, \dots, \lambda_n\}$.

Remark 7.26. In finite dimensional spaces $\lambda \in \sigma(T)$ corresponds to *both* $(\lambda I - T)$ not being injective *and* $(\lambda I - T)$ not being surjective. In V is infinite dimensional then just one of these may fail.

Proposition 7.27. *Suppose that V is a Banach space over $\mathbb{C}$ and that $T : V \to V$ is a bounded linear operator. Then $\sigma(T)$ is a compact (i.e. closed and bounded) set in $\mathbb{C}$. Furthermore, $\sigma(T) \subset \{z \in \mathbb{C} : |z| \leq \|T\|\}$.*

Proof. Claim: $\sigma(T) \subset \{z \in \mathbb{C} : |z| \leq \|T\|\}$. Suppose that $|\lambda| > \|T\|$; we shall show that $(\lambda I - T) : V \to V$ is invertible (so that $\lambda \notin \sigma(T)$). For $N \geq 1$, define

$$S_N = \sum_{n=1}^{N} \frac{T^{n-1}}{\lambda^n} \in B(V).$$

Write $\theta = \|T\|/|\lambda| < 1$. Since, for $1 \leq M \leq N$,

$$\|S_N - S_M\| = \left\| \sum_{n=M+1}^{N} \frac{T^{n-1}}{\lambda^n} \right\| \leq \sum_{n=M+1}^{N} \frac{1}{|\lambda|}\theta^{n-1} = \frac{1}{|\lambda|}\frac{\theta^M - \theta^N}{1-\theta},$$

we see that S_N is a Cauchy sequence. Since $B(V)$ is a Banach space, S_N converges to some $S \in B(V)$, as $N \to +\infty$. By direct claculation,

$$(\lambda I - T)S_N = S_N(\lambda I - T) = I - \lambda^{-N}T^N,$$

and letting $N \to +\infty$ gives that

$$(\lambda I - T)S = S(\lambda I - T) = I.$$

Thus $(\lambda I - T)$ is invertible and, in particular, $\lambda \notin \sigma(T)$.

Claim: $\sigma(T)$ is closed. Choose $\lambda \notin \sigma(T)$, then there exists $S \in B(V)$ such that $S(\lambda I - T) = (\lambda I - T)S = I$. Choose $\epsilon < 1/\|S\|$, then we shall show that

$$\{\mu \in \mathbb{C} : |\mu - \lambda| < \epsilon\} \cap \sigma(T) = \varnothing,$$

i.e., $\{\mu \in \mathbb{C} : |\mu - \lambda| < \epsilon\} \subset \mathbb{C} \backslash \sigma(T)$. If $|\mu - \lambda| < \epsilon$ then, since $(\lambda I - T)S = I$,

$$(\mu I - T) = (\lambda I - T) + (\mu - \lambda)I = (\lambda I - T)(I + (\mu - \lambda)S).$$

However,

(i) $(\lambda I - T)$ is invertible by assumption;

(ii) $(I + (\mu - \lambda)S)$ is invertible since

$$\|(\mu - \lambda)S\| = |\mu - \lambda|\|S\| < \epsilon\|S\| < 1.$$

So we have that the product $(\lambda I - T)(I + (\mu - \lambda)S)$ is invertible, i.e., $(\mu I - T)$ is invertible. This shows that $\mu \notin \sigma(T)$, as required. Therefore, $\mathbb{C} \backslash \sigma(T)$ is open and so $\sigma(T)$ is closed. □

Exercise 7.28. (See problem sheets.) Let $T \in B(V)$. Show that if $\|T\| < 1$ then $I - T$ is invertible. (Hint: Show that $I + T + \cdots + T^n$, $n \geq 1$ is a Cauchy sequence in $B(V)$ and hence has a limit.)

Remark 7.29. It is also true that $\sigma(T) \neq \varnothing$. To see this, one needs to consider the complex function

$$\mathbb{C} \to B(V) : \lambda \mapsto \sum_{n=1}^{\infty} \frac{T^{n-1}}{\lambda^n} = (\lambda I - T)^{-1}.$$

There is a theory of analytic functions of a complex variable which take values in Banach spaces. If $\sigma(T) = \varnothing$ then one can show that the above function is analytic and bounded. As for complex valued analytic (= differentiable = holomorphic) functions of a complex variable, this forces the function to be constant. (Recall Liouville's Theorem.) One can also show that $\|(\lambda I - T)^{-1}\| \to 0$, as $|\lambda| \to +\infty$, so the constant must be zero. Clearly, this is impossible.

Example. Let $V = \ell^p$, $1 \leq p < \infty$, and let T be the shift operator $T(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots)$. We know that $\|T\| = 1$, so $\sigma(T) \subset \{z \in \mathbb{C} : |z| \leq 1\}$. We will show that the set of eigenvalues of T is $\{z \in \mathbb{C} : |z| < 1\}$ and that $\sigma(T) = \{z \in \mathbb{C} : |z| \leq 1\}$. In particular, there are points in the spectrum of T which are not eigenvalues.

Suppose that λ is an eigenvalue of T. Then we must be able to find a non-zero $(x_1, x_2, x_3, \dots) \in \ell^p$ such that

$$(x_2, x_3, x_4, \dots) = T(x_1, x_2, x_3, \dots) = \lambda(x_1, x_2, x_3, \dots).$$

Looking at each co-ordinate in turn, we find the conditions

$$x_2 = \lambda x_1,\ x_3 = \lambda x_2 = \lambda^2 x_1,\ x_4 = \lambda x_3 = \lambda^3 x_1,\ \ldots,\ x_n = \lambda x_{n-1} = \lambda^{n-1} x_1, \ldots.$$

There is no condition on x_1, so let us choose $x_1 = 1$. Then $(1, \lambda, \lambda^2, \ldots)$ has the required property:

$$T(1, \lambda, \lambda^2, \ldots) = (\lambda, \lambda^2, \lambda^3, \ldots) = \lambda(1, \lambda, \lambda^2, \ldots).$$

Clearly, $(1, \lambda, \lambda^2, \ldots) \in \ell^p$ if and only if $|\lambda| < 1$. (Note that, apart from the choice of x_1, every eigenvector for λ has this form.)

Since eigenvalues are contained in the spectrum, this shows that

$$\{z \in \mathbb{C} : |z| < 1\} \subset \sigma(T).$$

Since $\sigma(T)$ is closed,

$$\{z \in \mathbb{C} : |z| \leq 1\} = \overline{\{z \in \mathbb{C} : |z| < 1\}} \subset \sigma(T).$$

On the other hand, $\|T\| = 1$, so, by Proposition 7.27,

$$\sigma(T) \subset \{z \in \mathbb{C} : |z| \leq 1\}.$$

Hence

$$\sigma(T) = \{z \in \mathbb{C} : |z| \leq 1\}.$$

Given $T : V \to V$, we can consider powers $T^2, T^3, \ldots, T^n, \ldots$. We can also form polynomial combinations: if $P(x) = a_n x^n + \cdots + a_1 x + a_0$ then we write $P(T) = a_n T^n + \cdots + a_1 T + a_0 I$.

Proposition 7.30. *If $P(x)$ is a polynomial then*

$$\sigma(P(T)) = \{P(\lambda) : \lambda \in \sigma(T)\}.$$

Proof. Suppose P has degree n. For a fixed $\lambda \in \mathbb{C}$, we can write

$$\lambda - P(z) = a(\beta_1 - z)(\beta_2 - z)\cdots(\beta_n - z), \tag{$*$}$$

where $\beta_1, \ldots, \beta_n \in \mathbb{C}$ are the roots of the polynomial $z \mapsto \lambda - P(z)$. We can then write

$$\lambda I - P(T) = a(\beta_1 I - T)(\beta_2 I - T)\cdots(\beta_n I - T).$$

If $\lambda \in \sigma(P(T))$ then $\lambda I - P(T)$ is not invertible, so $(\beta_i I - T)$ is not invertible for some i, giving $\beta_i \in \sigma(T)$. Substituting $z = \beta_i$ in $(*)$, we have $\lambda = P(\beta_i)$. This shows that $\sigma(P(T)) \subset \{P(\lambda) : \lambda \in \sigma(T)\}$.

Now suppose that $\lambda \notin \sigma(P(T))$, so that $\lambda I - P(T)$ is invertible with inverse S, say. For each $i = 1, \ldots, n$, we have (remembering that everything commutes)

$$\lambda I - P(T) = (\beta_i I - T)Q_i(T),$$

for some polynomial Q_i. Then

$$I = S(\lambda I - P(T)) = S(\beta_i I - T)Q_i(T) = (\beta_i I - T)SQ_i(T) = SQ_i(T)(\beta_i I - T),$$

so $(\beta_i I - T)$ is invertible with inverse $SQ_i(T)$. Thus $\{\beta_1, \ldots, \beta_n\} \cap \sigma(T) = \varnothing$. Since the equation $\lambda - P(z) = 0$ has no other solutions, this shows that $\sigma(P(T))^c \cap \{P(\lambda) : \lambda \in \sigma(T)\} = \varnothing$. This completes the proof.

□

Since $\sigma(T) \subset \mathbb{C}$ is bounded, we can make the following definition.

Definition 7.31. We define the *spectral radius* of T to be the number

$$\rho(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}.$$

By Proposition 7.27, we know that $\rho(T) \leq \|T\|$. A stronger result is given below.

Theorem 7.32 (Spectral Radius Theorem)**.**

$$\rho(T) = \lim_{n\to+\infty} \|T^n\|^{1/n}.$$

Idea. Since $\sigma(T^n) = \{\lambda^n : \lambda \in \sigma(T)\}$, for each $n \geq 1$, we have

$$\rho(T)^n = \rho(T^n) \leq \|T^n\|,$$

i.e., $\rho(T) \leq \|T^n\|^{1/n}$. Therefore, $\rho(T) \leq \liminf_{n\to+\infty} \|T^n\|^{1/n}$.

The proof of the lower bound is omitted. (It requires the theory of analytic Banach space valued functions.)

□

7.3 Spectra of Operators on Hilbert Spaces

Let $T : H \to H$ be a bounded linear operator on a Hilbert space H and let $T^* : H \to H$ be its adjoint operator.

Lemma 7.33. *(i) $\sigma(T^*) = \overline{\sigma(T)} = \{\overline{\lambda} : \lambda \in \sigma(T)\}$, i.e. $\lambda \in \sigma(T) \iff \bar{\lambda} \in \sigma(T^*)$;*

(ii) if T^{-1} exists then $\sigma(T^{-1}) = (\sigma(T))^{-1} = \{\lambda^{-1} : \lambda \in \sigma(T)\}$, i.e. $\lambda \in \sigma(T) \iff \lambda^{-1} \in \sigma(T^{-1})$.

Proof. (i) We have $\lambda \notin \overline{\sigma(T)}$ if and only if $S = (\bar{\lambda}I - T)$ is invertible. Since $SS^{-1} = I \iff I = I^* = (SS^{-1})^* = (S^{-1})^*S^*$ and $S^{-1}S = I \iff I = I^* = (S^{-1}S)^* = S^*(S^{-1})^*$, $S^* = (\lambda I - T^*)$ is invertible if and only if $(\bar{\lambda}I - T)$ is invertible.

(ii) If $\lambda \notin \sigma(T)$ then $S(\lambda I - T) = I = (\lambda I - T)S$, where $S = (\lambda I - T)^{-1}$. Notice that $ST = -I + \lambda S = TS$. Thus if $U = -\lambda ST = -\lambda TS$ then $U(\lambda^{-1}I - T^{-1}) = I = (\lambda^{-1}I - T^{-1})U$, so $(\lambda^{-1}I - T^{-1})$ is invertible, i.e., $\lambda^{-1} \notin \sigma(T^{-1})$. By symmetry, $\sigma(T^{-1}) = \{\lambda^{-1} : \lambda \in \sigma(T)\}$. □

Definition 7.34. We say that $T : H \to H$ is *normal* if $TT^* = T^*T$. (Notice that if T is self-adjoint then T is normal.)

Suppose that $T^*T = I$. Then (and only then) we have, for all $x \in H$,

$$\|x\|^2 = \langle x, x\rangle = \langle x, T^*Tx\rangle = \langle Tx, Tx\rangle = \|Tx\|^2.$$

So T is an isometry if and only if $T^*T = I$.

Definition 7.35. We say that T is unitary if $T^* = T^{-1}$. (Note that if T is unitary then T is an isometry. Conversely, if T is an isometry and T is invertible then T is unitary.)

Theorem 7.36. *(i) If T is normal then $\rho(T) = \|T\|$.*

(ii) If T is an isometry then $\rho(T) = 1$.

(iii) If T is unitary then $\sigma(T) \subset \{z \in \mathbb{C} : |z| = 1\}$.

(iv) It T is self-adjoint then $\rho(T) = \|T\|$ and $\sigma(T) \subset [-\|T\|, \|T\|] \subset \mathbb{R}$.

Proof. (i) For $n > 0$,

$$\begin{aligned}\|T^{2^n}\|^2 &= \|(T^{2^n})^*(T^{2^n})\| = \|(T^*T)^{2^n}\| \\ &\text{(by normality } T^*T = TT^*) \\ &= \|(T^*T)^{2^{n-1}}\|^2 \\ &\text{(since } S = (T^*T)^{2^{n-1}} \text{ satisfies } \|S^*S\| = \|S\|^2) \\ &= \|(T^*T)^{2^{n-2}}\|^4 = \cdots = \|T^*T\|^{2^n} = \|T\|^{2^{n+1}}.\end{aligned}$$

So $\|T^{2^n}\| = \|T\|^{2^n}$. Thus $\rho(T) = \lim_{n\to+\infty} \|T^{2^n}\|^{1/2^n} = \|T\|$.

(ii) We have $\|T^n\|^2 = \|(T^*)^n T^n\|$. Repeatedly using $T^*T = I$, we get $\|T^n\|^2 = \|I\| = 1$. Hence $\rho(T) = \lim_{n\to+\infty} \|T^n\|^{1/n} = 1$.

(iii) If T is unitary then T is an isometry, so by (ii), $\sigma(T) \subset \{z \in \mathbb{C} : |z| \le 1\}$. Thus by Lemma 7.33, $\sigma(T^{-1}) \subset \{z \in \mathbb{C} : |z| \ge 1\}$. But $\sigma(T^{-1}) = \sigma(T^*) = \{\bar{z} : z \in \sigma(T)\}$ and $|\bar{z}| = |z|$, so this forces $\sigma(T) \subset \{z \in \mathbb{C} : |z| = 1\}$.

(iv) If T is self-adjoint then T is normal so, by (i), $\rho(T) = \|T\|$. We already know that $\sigma(T) \subset \{z \in \mathbb{C} : |z| \le \|T\|\}$, so to complete the proof we just need to show that the spectrum is real. It will suffice to show that $(\lambda I - T)$ is invertible whenever $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Choose a real number r such that $0 < r < 1/\|T\|$. Then $(I + irT)^{-1}$ exists (since $\|irT\| = r\|T\| < 1$). Let $U = (I - irT)(I + irT)^{-1}$. Then we also have $U = (I + irT)^{-1}(I - irT)$. *Claim: U is unitary.* We shall use the facts that $(S^*)^{-1} = (S^{-1})^*$ and $(zS)^* = \bar{z}S^*$. We have

$$\begin{aligned}U^{-1} &= (I + irT)(I - irT)^{-1} \\ &= (I + irT^*)(I - irT^*)^{-1} \\ &= (I - irT)^*((I + irT)^*)^{-1} \\ &= (I - irT)^*((I + irT)^{-1})^* \\ &= ((I + irT)^{-1}(I - irT))^* = U^*,\end{aligned}$$

as required.

Now, if $\lambda \in \mathbb{C} \setminus \mathbb{R}$ then

$$\left|\frac{1 - ir\lambda}{1 + ir\lambda}\right| \ne 1,$$

so, by (iii),

$$\left(\frac{1-ir\lambda}{1+ir\lambda}\right) I - U$$

is invertible. However,

$$\begin{aligned}
&\left(\frac{1-ir\lambda}{1+ir\lambda}\right) I - U \\
&= \frac{1}{1+ir\lambda}\left((1-ir\lambda)(I+irT) - (I-irT)(1+ir\lambda)\right)(I+irT)^{-1} \\
&= \frac{-2ir}{1+ir\lambda}(\lambda I - T)(I+irT)^{-1}.
\end{aligned}$$

Since the L.H.S. is invertible and $(I+irT)^{-1}$ is invertible, this shows that $(\lambda I - T)$ is invertible, as required. $\square$

8 Compact Operators

8.1 Definition and Properties

As we have seen the spectrum of a linear operator on an infinite dimensional space can be very different from that of a matrix. In this section, we shall consider, in contrast, a special class of operators whose spectral theory is very similar to the finite dimensional case. There are called *compact operators*.

Definition 8.1. Let V and V' be normed vector spaces. A linear operator $T : V \to V'$ is *compact* if, for any bounded sequence $x_n \in V$, the sequence $Tx_n \in V'$ contains a convergent subsequence. The set of compact linear operators from V to V' is denoted $K(V, V')$.

Theorem 8.2. *(i) $K(V, V') \subset B(V, V')$ (i.e. compact operators are automatically bounded).*

(ii) $K(V, V')$ is a vector space and, if V' is a Banach space, then $K(V, V')$ is a closed subspace of $B(V, V')$.

(iii) If $T, S \in B(V, V')$ and at least one of them is compact then TS is compact.

Proof. (i) Suppose that a linear operator T is not bounded. Then, by definition, there exists a sequence $x_n \in V$, with $\|x_n\| = 1$, such that $\|Tx_n\| \to +\infty$, as $n \to +\infty$. Clearly, the sequence x_n is bounded. Moreover, any subsequence of Tx_n will also have norm tending to $+\infty$ and so cannot be convergent. Thus T is not compact.

(ii) The first statement is an easy exercise. Now suppose that V' is a Banach space and that $T_n \in K(V, V')$ converges to $T \in B(V, V')$ (in the $\|\cdot\|$-topology on $B(V, V')$). Let x_j be a bounded sequence in V. Since T_1 is compact, it has a subsequence $x_j^{(1)}$ such that $T_1 x_j^{(1)}$ converges, as $j \to +\infty$. As $x_j^{(1)}$ is itself a bounded sequence and T_2 is compact, it has a subsequence $x_j^{(2)}$ such that $T_2 x_j^{(2)}$ converges, as $j \to +\infty$. Continuing inductively, for each m, we have a sequence $x_j^{(m)}$ which is a subsequence of $x_j^{(m-1)}$ such that $T_k x_j^{(m)}$ converges as $j \to +\infty$, for all $k \leq m$. Now consider the sequence $y_k = x_k^{(k)}$. For each m, from some term on, y_k is a subsequence of $x_k^{(m)}$. Hence $T_n y_k$ converges, as $k \to +\infty$, for all n. We have

$$
\begin{aligned}
\|Ty_k - Ty_l\| &\leq \|Ty_k - T_n y_k\| + \|T_n y_k - T_n y_l\| + \|T_n y_l - Ty_l\| \\
&\leq \|T - T_n\|(\|y_k\| + \|y_l\|) + \|T_n y_k - T_n y_l\|
\end{aligned}
$$

Given $\epsilon > 0$, for n sufficiently large, we have $\|T - T_n\| < \epsilon$ and, for k, l sufficiently large, we have $\|T_n y_k - T_n y_l\| < \epsilon$ (because $T_n y_j$ converges as $j \to +\infty$). Since $\|y_k\| + \|y_l\|$ is bounded, this shows that Ty_j is a Cauchy sequence and so, because V' is a Banach space, it converges. Hence T is compact.

(iii) Let x_n be a bounded sequence in V. If S is compact then there is a subsequence x_{n_i} such that Sx_{n_i} converges. Since T is bounded (and hence continuous), $T(Sx_{n_i})$ also converges and so TS is compact. If S is merely bounded then the sequence Sx_n is bounded. If T is compact then there is a subsequence Sx_{n_i} such that $T(Sx_{n_i})$ converges, so again TS is compact. $\square$

The next result makes it clearer why these operators are called compact.

Theorem 8.3. *Let V and V' be normed vector spaces and let $T : V \to V'$ be a linear operator. Then T is compact if and only if the image of any bounded subset of V is relatively compact, i.e. for any bounded set $B \subset V$, $\overline{T(B)}$ is compact.*

Proof. ($\Longrightarrow$) Suppose that T is compact and let $B \subset V$ be bounded. Let $y_n \in \overline{T(B)}$ be a sequence. For each n, choose $x_n \in B$ such that

$$\|y_n - Tx_n\| < \frac{1}{n}.$$

Then the sequence x_n is bounded and so, since T is compact, Tx_n has a convergent subsequence Tx_{n_j}, converging to y, say. By construction, $y \in \overline{T(B)}$. Since

$$\|y_{n_j} - Tx_{n_j}\| < \frac{1}{n_j},$$

we have that y_{n_j} converges to y also. So y_n has a convergent subsequence and $\overline{T(B)}$ is compact.

($\Longleftarrow$) Now suppose that, whenever $B \subset V$ is bounded, $\overline{T(B)}$ is compact. Let x_n be a bounded sequence in V. This is itself a bounded subset of V, so $\overline{\{Tx_n\}}$ is compact. But Tx_n is a sequence in this compact set, so it has a convergent subsequece. Hence T is compact. $\square$

That compact operators are rather special is shown by the next result.

Theorem 8.4. *Let V be an infinite dimensional normed vector space.*

(i) The identity operator $I : V \to V$ is not compact.

(ii) If $T : V \to V$ is invertible then it is not compact.

Proof. (i) Since V is infinite dimensional, the closed unit ball $B = \{x \in V : \|x\| \leq 1\}$ is not compact. Since $I(B) = B$ (closed), this shows that I is not compact.

Suppose that T is invertible. If T were, in addition, compact, we would have that $I = TT^{-1}$ is compact, contradicting (i). $\square$

Example 8.5. Let $k \in L^2([a, b] \times [a, b], \mathbb{C}$. Then $K : L^2([a, b], \mathbb{C}) \to L^2([a, b], \mathbb{C})$ defined by

$$(Kf)(t) = \int_a^b k(s, t) f(s)\, ds$$

is a compact operator. (See exercises.)

8.2 Spectral Theory of Compact Operators

We will now consider the spectral theory of compact operators. In fact, we will restrict to compact self-adjoint operators on a Hilbert space, as this is the setting in which the strongest results hold. Throughout this section, H will be a Hilbert space over $\mathbb{C}$.

We will write $\sigma_p(T)$ for the set of eigenvalues of T, also called the point spectrum. Of course, $\sigma_p(T) \subset \sigma(T)$.

Lemma 8.6. *Let $T : H \to H$ be a compact self-adjoint operator. Then at least one of $\|T\|$ and $-\|T\|$ belongs to $\sigma_p(T)$.*

Proof. If $T = 0$ the lemma is trivial. So suppose that $T \neq 0$. Then

$$\|T\| = \sup_{\|x\|=1} |\langle Tx, x\rangle|$$

and so there is a sequence $x_n \in H$, with $\|x\| = 1$, such that $|\langle Tx_n, x_n\rangle| \to \|T\|$, as $n \to +\infty$. Since T is compact, there is a subsequence x_{n_j} such that Tx_{n_j} converges to $y \in H$, say. Then

$$\lim_{j\to+\infty} \langle Tx_{n_j}, x_{n_j}\rangle = \pm\|T\| = \alpha \neq 0.$$

We have

$$\begin{aligned}
\|Tx_{n_j} - \alpha x_{n_j}\|^2 &= \|Tx_{n_j}\|^2 + \alpha^2\|x_{n_j}\|^2 - 2\alpha\langle Tx_{n_j}, x_{n_j}\rangle \\
&\le \alpha^2 + \alpha^2 - 2\alpha\langle Tx_{n_j}, x_{n_j}\rangle \\
&= 2\alpha(\alpha - \langle Tx_{n_j}, x_{n_j}\rangle) \to 0,
\end{aligned}$$

as $j \to +\infty$. Combining $Tx_{n_j} \to y$ and $Tx_{n_j} - \alpha x_{n_j} \to 0$ gives $\lim_{j\to+\infty} \alpha x_{n_j} = y$, i.e. $\lim_{j\to+\infty} x_{n_j} = y/\alpha$. Since T is bounded (and hence continuous), we deduce $Tx_{n_j} \to (Ty)/\alpha$. Thus, $Ty = \alpha y$ and $\|y\| \neq 0$ (because $\|x_{n_j}\| = 1$). Therefore, $\alpha \in \sigma_p(T)$. □

Theorem 8.7 (Spectral Theorem for compact self-adjoint operators)**.** *Let $T : H \to H$ be a compact self-adjoint operator. Then exactly one of the following holds.*

(a) If $\dim \operatorname{Range}(T) = N$ then T has N non-zero real eigenvalues $\lambda_1, \dots, \lambda_n$ (not necessarily distinct). If $\dim H > N$ then 0 is also an eigenvalue. Moreover, there exist an orthonormal set $e_1, \dots, e_N$ of eigenvectors, corresponding to the λ_k, such that, for all $x \in H$,

$$Tx = \sum_{k=1}^{N} \lambda_k \langle x, e_k\rangle e_k.$$

(b) If $\dim \operatorname{Range}(T) = \infty$ then T has countably many non-zero real eigenvalues $\lambda_1, \lambda_2, \dots,$ such that $\lim_{k\to+\infty} \lambda_k = 0$. Moreover, there exists an orthonormal set of eigenvectors e_k, corresponding to the λ_k, such that, for all $x \in H$,

$$Tx = \sum_{k=1}^{\infty} \lambda_k \langle x, e_k\rangle e_k.$$

Any eigenspace corresponding to non-zero eigenvalues is finite dimensional.

Proof. For the first part of the proof the range may be finite or infinite dimensional. We construct the family e_k as follows. Set $H_1 = H$ and $T_1 = T|_{H_1} : H_1 \to H_1$. Then, by Lemma 8.6, there exists $\lambda_1 \in \sigma_p(T)$ such that $|\lambda_1| = \|T_1\| = \|T\|$. Let e_1 be the associated eigenvector normalised to have $\|e_1\| = 1$. Set $H_2 = (\mathbb{C}e_1)^\perp$. Then $H_2 \subset H_1$ and if $x \in H_2$ then

$$\langle Tx, e_1\rangle = \langle x, Te_1\rangle = \lambda_1\langle x, e_1\rangle = 0,$$

so $Tx \in H_2$. Let $T_2 = T|_{H_2} : H_2 \to H_2$; this is compact and self-adjoint. Then there exists $\lambda_2 \in \sigma_p(T_2)$ such that $\|\lambda_2\| = \|T_2\|$. Let e_2 be the associated normalised eigenvector and set $H_3 = (\mathbb{C}e_1 \oplus \mathbb{C}e_2)^\perp$. Continue in this way, building up a family of spaces

$$\cdots H_n \subset H_{n-1} \subset \cdots \subset H_1 = H,$$

with

$$H_n = (\mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_{n-1})^\perp,$$

$\lambda_n \in \sigma_p(T_n)$ such that $|\lambda_n| = \|T_n\|$, and e_n the associated normalised eigenvector of $T_n = T|_{H_n} : H_n \to H_n$. We see that there are two cases.

(a) If $\dim \operatorname{Range}(T) = N$ then $T_{N+1} = T|_{H_{N+1}} = 0$. If $\dim H = N$ then $H_{N+1} = \{0\}$. if $\dim H > N$ then $H_{N+1} = \ker T$, i.e. H_{N+1} contains the eigenvectors of T corresponding to the eigenvalue 0.

For $x \in H$, consider

$$y = x - \sum_{k=1}^{N}\langle x, e_k\rangle e_k.$$

Then

$$Ty = Tx - \sum_{k=1}^{N}\langle x, e_k\rangle Te_k = Tx - \sum_{k=1}^{N}\lambda_k\langle x, e_k\rangle e_k.$$

We have $\langle y, e_k\rangle = 0$, $k = 1, \ldots, N$, so $y \in H_{N+1}$ Hence $Ty = T|_{H_{N+1}}y = T_{N+1}y = 0$. This gives

$$Tx = \sum_{k=1}^{N}\lambda_k\langle x, e_k\rangle e_k,$$

as required.

(b) If $\dim \operatorname{Range}(T) = \infty$ the construction continues with $T_N \neq 0$ for all N. For $x \in H$ and $N \geq 1$, consider

$$y_N = x - \sum_{k=1}^{N-1}\langle x, e_k\rangle e_k.$$

Then $y_N \in H_N$ and so

$$\|x\|^2 = \|y_N\|^2 + \sum_{k=1}^{N-1}|\langle x, e_k\rangle|^2,$$

giving $\|x\| \geq \|y_N\|$ for all $N \geq 1$. By construction, $\|T_N\| = |\lambda_N|$, so

$$\left\|Tx - \sum_{k=1}^{N-1}\lambda_k\langle x, e_k\rangle e_k\right\| = \|Ty_N\| \leq \|T_N\|\|y_N\| = |\lambda_N|\|y_N\| \leq |\lambda_N|\|x\|.$$

Thus the formula
$$Tx = \sum_{k=1}^{\infty} \lambda_k \langle x, e_k \rangle e_k.$$
will follow once we show that $\lim_{k\to+\infty} \lambda_k = 0$.

For a contradiction, suppose that there exists a subsequence λ_{k_i} and $\epsilon > 0$ such that $|\lambda_{k_i}| \geq \epsilon$ for all $i \geq 1$. Then, for all $i, j \geq 1$,
$$\begin{aligned}\|Te_{k_i} - Te_{n_j}\|^2 &= \langle Te_{k_i} - Te_{k_j}, Te_{k_i} - Te_{k_j} \rangle = \langle \lambda_{k_i} e_{k_i} - \lambda_{k_j} e_{k_j}, \lambda_{k_i} e_{k_i} - \lambda_{k_j} e_{k_j} \rangle \\ &= |\lambda_{k_i}|^2 + |\lambda_{k_j}|^2 \geq 2\epsilon^2.\end{aligned}$$

As a consequence, any subsequence of Te_{k_i} fails the Cauchy criterion and does not converge. But the sequence e_{k_i} is bounded, so this contradicts the compactness of T.

Now we show that the λ_k constructed above are the only non-zero real eigenvalues. Suppose that λ is an eigenvalue that does not belong to this sequence and let the associated normalised eigenvector be e. Then, for each k,
$$\lambda \langle e, e_k \rangle = \langle \lambda e, e_k \rangle = \langle Te, e_k \rangle = \langle e, Te_k \rangle = \langle e, \lambda_k e_k \rangle = \lambda_k \langle e, e_k \rangle.$$

By hypothesis, $\lambda \neq \lambda_k$ for all k, so we have that $\langle e, e_k \rangle = 0$ for all k. Then
$$\lambda e = Te = \sum_{k=1}^{\infty} \lambda_k \langle e, e_k \rangle e_k = 0,$$
so $\lambda = 0$.

To complete the proof, we show that the eigenspace associated to any non-zero eigenvalue is finite dimensional. Let $\lambda \in \sigma_p(T)$, $\lambda \neq 0$ and let E_λ denote the associated eigenspace. Let $\widetilde{T} = T|_{E_\lambda}$. If $x \in E_\lambda$ then $Tx = \lambda x$, so $\widetilde{T} : E_\lambda \to E_\lambda$. Since $\widetilde{T}x = \lambda x$, $\widetilde{T}$ maps the closed unit ball into the closed ball of radius $|\lambda|$. But since $\widetilde{T}$ is compact, $\{x \in E_\lambda : \|x\| \leq |\lambda|\}$ is compact, which implies that E_λ is finite dimensional. □

Corollary 8.8. *Let H be an infinite dimensional separable Hilbert space and let $T : H \to H$ be a compact self-adjoint operator. Then there exists and orthonormal basis $\{e_k\}_{k=1}^{\infty}$ of H such that $Te_k = \lambda_k e_k$, for all $k \geq 1$, and, for all $x \in H$,*
$$Tx = \sum_{k=1}^{\infty} \lambda_k \langle x, e_k \rangle e_k.$$

Proof. This follows immediately from Theorem 8.7. □

We remark that since the proof that $\dim E_\lambda$ is finite only uses the compactness of T, we have the following result.

Proposition 8.9. *Let $T : H \to H$ be compact. Then for all non-zero eigenvalues, the associated eigenspace is finite dimensional.*

Proposition 8.10. *Let H be an infinite dimensional Hilbert space and let $T : H \to H$ be a compact self-adjoint operator. Then $\sigma(T) = \overline{\sigma_p(T)}$.*

Proof. We will only give the proof in the case where H is separable. By Corollary 8.8, for any $x \in H$,

$$Tx = \sum_{k=1}^{\infty} \lambda_k \langle x, e_k \rangle e_k,$$

where $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for H. Since $x = \sum_{k=1}^{\infty} \langle x, e_k \rangle e_k$, we have

$$(T - \mu I)x = \sum_{k=1}^{\infty} (\lambda_k - \mu) \langle x, e_k \rangle e_k.$$

Let $\mu \in \mathbb{C} \backslash \overline{\sigma_p(T)}$, which is an open subset of $\mathbb{C}$. Hence, there exists $\epsilon > 0$ such that $|\mu - \lambda| > \epsilon$ for all $\lambda \in \sigma_p(T) \subset \overline{\sigma_p(T)}$. Consider the operator S defined by

$$Sy = \sum_{k=1}^{\infty} \frac{\langle y, e_k \rangle}{\lambda_k - \mu} e_k.$$

Since $|\lambda_k - \mu| > \epsilon$ the series converges and

$$\|Sy\|^2 = \sum_{k=1}^{\infty} \left| \frac{\langle y, e_k \rangle}{\lambda_k - \mu} \right|^2 \leq \epsilon^{-2} \sum_{k=1}^{\infty} |\langle y, e_k \rangle|^2 = \epsilon^{-2} \|y\|^2.$$

In particular, S is bounded with $\|S\| \leq \epsilon^{-1}$.

Since

$$(T - \mu I)Sy = \sum_{k=1}^{\infty} (\lambda_k - \mu) \langle Sy, e_k \rangle e_k = \sum_{k=1}^{\infty} \frac{\lambda_k - \mu}{\lambda_k - \mu} \langle y, e_k \rangle e_k = y$$

and

$$S(T - \mu I)x = \sum_{k=1}^{\infty} \frac{\langle (T - \mu I)x, e_k \rangle}{\lambda_k - \mu} e_k = \sum_{k=1}^{\infty} \frac{\lambda_k - \mu}{\lambda_k - \mu} \langle x, e_k \rangle e_k = x.$$

we have $S = (T - \mu I)^{-1}$ and $\mu \in \mathbb{C} \backslash \sigma(T)$. Hence $\sigma(T) \subset \overline{\sigma_p(T)}$. The reverse inclusion follows from the fact that $\sigma_p(T) \subset \sigma(T)$ with $\sigma(T)$ closed. $\square$

9 Sturm-Liouville problems

9.1 Second order linear differential operators

By a second order linear differential operator we mean one of the form

$$Lu = a_2(x)\frac{d^2u}{dx^2} + a_1(x)\frac{du}{dx} + a_0(x)u,$$

where a_0, a_1, a_2 are continuous on an interval $[a, b]$, with a_2 never zero. For the moment, we will not worry about the space in which u lies. By multiplying by the integrating factor

$$\frac{1}{a_2(x)} \exp \int \frac{a_1(x)}{a_2(x)}\, dx,$$

we may reduce this to

$$Lu = \frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u(x)$$

and this is the form we will study.

9.2 The Sturm-Liouville problem

A Sturm-Liouville problem is a differential equation of the form

$$Lu := \frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u(x) = \lambda u(x) \qquad \text{with } u(a) = u(b) = 0,$$

where p and q are given functions on the interval $[a, b]$. The values for which the problem has a non-trivial solution are called the eigenvalues of the Sturm-Liouville problem and the corresponding u are called eigenfunctions. An eigenvalue is called simple if the corresponding eigenspace is one-dimensional.

Theorem 9.1 (Sturm-Liouville Theorem)**.** *If $p \in C^1([a, b], \mathbb{R})$ with $p(x) > 0$ for all $x \in [a, b]$, $q \in C([a, b], \mathbb{R})$, and 0 is not an eigenvalue of the Sturm-Liouville problem, then*

(i) the eigenvalues λ_n of the Sturm-Liouville problem satisfy $|\lambda_n| \to +\infty$, as $n \to +\infty$;

(ii) the eigenvalues are simple;

(iii) the eigenfunctions form an orthogonal basis for $L^2([a, b], \mathbb{R})$.

Here $C^1([a, b], \mathbb{R})$ denotes the space of differentiable functions $f : [a, b] \to \mathbb{R}$ such that f' is continuous.

Remark 9.2. The requirement that zero is not an eigenvalue of the problem is a technical condition used to ensure that the equation $Lu = f$, $u(a) = u(b) = 0$, has a unique solution. (See below.)

This result is approached via the spectral theory of linear operators. L is defined as a linear operator on a subspace $\mathcal{D}$ of $L^2([a,b],\mathbb{R})$: $\mathcal{D}$ is the set of equivalence classes $[u]$ in $L^2([a,b],\mathbb{R})$ such that

- there is a representative u of $[u]$ such u is differentiable,
- pu' is differentiable almost everywhere and $(pu')' \in L^2([a,b],\mathbb{R})$,
- $u(a) = u(b) = 0$.

In fact, $\mathcal{D}$ is dense in $L^2([a,b],\mathbb{R})$ with respect to $\|\cdot\|_2$.

Remark 9.3. The fact that the Sturm-Liouville eigenvalures are unbounded shows that L is not a bounded operator.

The eigenfunctions of the Sturm-Liouville problem are the eigenvectors of L and we will use the term eigenfunction. In order to prove Theorem 9.1, we will show that $L : \mathcal{D} \to L^2([a,b],\mathbb{R})$ is "almost" the inverse of a compact self-adjoint operator $K : L^2([a,b],\mathbb{R}) \to L^2([a,b],\mathbb{R})$ and apply the results of the preceding section.

9.3 The differential equation $Lu = f$

We begin with a couple of lemmas. They refer to (non-zero) solutions of the equation $Lu = 0$ but the boundary condition $u(a) = u(b) = 0$ is *not* imposed so the "no zero eigenvalue" condition is not violated.

Lemma 9.4. *If both u_1 and u_2 satisfy the equation $Lu = 0$ then*

$$\Delta(x) = p(x)(u_2'(x)u_1(x) - u_1'(x)u_2(x))$$

is constant, Δ say. Furthermore, if $\Delta \neq 0$ then u_1 and u_2 are linearly independent.

Proof. Differentiating $\Delta(x)$ with respect to x and using $pu'' = -p'u' + qu$, we obtain

$$\begin{aligned}\Delta' &= p'(u_2'u_1 - u_1'u_2) + p(u_2''u_1 - u_1''u_2)\\ &= p'(u_2'u_1 - u_1'u_2) + ((-p'u_2' + qu_2)u_1 - (-p'u_1' + qu_1)u_2) = 0.\end{aligned}$$

Therefore $\Delta(x) = \Delta$ is constant.

Now suppose that u_1 and u_2 are linearly dependent. Then there are constants α_1, α_2, not both zero, such that $\alpha_1 u_1 + \alpha_2 u_2 = 0$. Without loss of generality, suppose $\alpha_2 \neq 0$. Then $u_2 = -\alpha_1 u_1/\alpha_2$ and $u_2' = -\alpha_2 u_1'/\alpha_2$. Substituting these identities into the definition of $\Delta(x)$ gives $\Delta = 0$. The required statement is the contrapositive. $\square$

Lemma 9.5. *The equation $Lu = 0$ has two linearly independent solutions u_1, u_2 such that $u_1(a) = u_2(b) = 0$.*

Proof. From the elementary theory of ODEs, there is a unique function u_1 such that $Lu_1 = 0$, $u_1(a) = 0$ and $u_1'(a) = 1$. (One then easily sees that every solution of $Lu = 0$, $u(a) = 0$ is a scalar multiple of u.) Similarly, there is a unique function u_2 such that $Lu_2 = 0$, $u_2(b) = 0$ and $u_2'(b) = 1$. To see they are linearly independent, consider

$$\Delta = \Delta(a) = p(a)(u_2'(a)u_1(a) - u_1'(a)u_2(a)) = -p(a)u_2(a).$$

If $\Delta = 0$ then $u_2(a) = 0$, so that $Lu_2 = 0$ with $u_2(a) = u_2(b) = 0$, i.e. zero is an eigenvalue of the Sturm-Liouville problem. Since zero is assumed not to be an eigenvalue, we have $\Delta \neq 0$ and so, by Lemma 9.4, u_1 and u_2 are linearly independent. □

Theorem 9.6. *The equation $Lu = f$, with the boundary conditions $u(a) = u(b) = 0$, has a solution of the form*

$$u(x) = \int_a^b k(x, y) f(y)\, dy,$$

where $k \in C([a, b] \times [a, b], \mathbb{R})$.

Proof. Set

$$k(x, y) = \begin{cases} u_1(x)u_2(y)/\Delta & \text{if } a \leq x \leq y \leq b \\ u_1(y)u_2(x)/\Delta & \text{if } a \leq y < x \leq b, \end{cases}$$

where u_1 and u_2 are the linearly independent functions obtained in Lemma 9.5. It is clear that k is continuous.

To complete the proof, we just need to check that

$$u(x) = \int_a^b k(x, y) f(y)\, dy,$$

is indeed a solution. First note that when $x = a$ we have $x \leq y$ throughout the domain of integration and so

$$u(a) = \frac{1}{\Delta} \int_a^b u_1(a)u_2(y)f(y)\, dy = 0.$$

By a similar argument, $u(b) = 0$.

We now substitute into the equation (for convenience multiplying by Δ). Since $u_1(x)$ and $u_2(x)$ can be taken outside the integration, we obtain

$$\Delta u(x) = \Delta \int_a^b k(x, y) f(y)\, dy = u_2(x) \int_a^x u_1(y)f(y)\, dy + u_1(x) \int_x^b u_2(y)f(y)\, dy,$$

$$\begin{aligned}(\Delta u(x))' &= u_2(x)u_1(x)f(x) + u_2'(x) \int_a^x u_1(y)f(y)\, dy - u_1(x)u_2(x)f(x) + u_1'(x) \int_x^b u_2(y)f(y)\, dy \\ &= u_2'(x) \int_a^x u_1(y)f(y)\, dy + u_1'(x) \int_x^b u_2(y)f(y)\, dy,\end{aligned}$$

and

$$\begin{aligned}(p(x)(\Delta u(x))')' &= (pu_2')' \int_a^x u_1(y)f(y)\,dy + pu_2'u_1 f + (pu_1')' \int_x^b u_2(y)f(y)\,dy - pu_1'u_2 f \\ &= (pu_2')' \int_a^x u_1(y)f(y)\,dy + (pu_1')' \int_x^b u_2(y)f(y)\,dy + \Delta f.\end{aligned}$$

Therefore,

$$\begin{aligned}(p(\Delta u)')' + q\Delta u &= ((pu_2')' + qu_2) \int_a^x u_1(y)f(y)\,dy + ((pu_1')' + qu_1) \int_x^b u_2(y)f(y)\,dy + \Delta f \\ &= (Lu_2) \int_a^x u_1(y)f(y)\,dy + (Lu_1) \int_x^b u_2(y)f(y)\,dy + \Delta f \\ &= \Delta f,\end{aligned}$$

as required. (We have assumed results about differentiating integrals.) □

Remark 9.7. The function k is called the Green's function for the problem.

9.4 An integral operator

Define an operator $K : L^2([a,b],\mathbb{R}) \to L^2([a,b],\mathbb{R})$ by

$$(Kf)(x) = \int_a^b k(x,y)f(y)\,dy.$$

Since k is continuous, K is compact. Since k is real valued, it is also easy to see that K is self-adjoint.

If $u \in \mathcal{D}$ and $f = Lu$ then Theorem 9.6 shows that $u = Kf = K(Lu)$, i.e. AL acts as the identity on $\mathcal{D}$. On the other hand, it follows from the proof of Theorem 9.6 that, if $f \in L^2([a,b],\mathbb{R})$, then $Kf \in \mathcal{D}$ (by virtue of being a solution of the equation) and that $LKf = f$. Thus $LK = I$.

Note that L fails to be the inverse of K as it is not defined on the whole of $L^2([a,b],\mathbb{R})$, only on the dense subset $\mathcal{D}$. Indeed, since K is compact, it cannot be invertible.

Lemma 9.8. *(i) The operator does not have zero as an eigenvalue.*
(ii) μ is an eigenvalue of K if and only if $\mu = \lambda^{-1}$ is an eigenvalue for L with the boundary condition $u(a) = u(b) = 0$.

Proof. (i) Suppose that $Kf = 0$ for some $f \in L^2([a,b],\mathbb{R})$. Then, for any $g \in L^2([a,b],\mathbb{R})$,

$$0 = \langle Kf, g\rangle = \langle f, Kg\rangle,$$

which implies that $f = 0$ (because u is orthogonal to the dense set $\mathcal{D} = \mathrm{Range}(K)$). Thus, zero is not an eigenvalue for K.

(ii) Let $u \in L^2([a,b],\mathbb{R})$ be an eigenfunction of K, i.e. $Ku = \mu u$. By part (i), $\mu \neq 0$, so we can write $u = \mu^{-1}Ku \in \mathcal{D}$ and $u(a) = u(b) = 0$. Applying L, we have $Lu = \mu^{-1}LKu = \mu^{-1}u$.

Conversly, let $u \in \mathcal{D}$, with $u(a) = u(b) = 0$, be an eigenvalue for L, i.e. $Lu = \lambda u$. Applying K we get $u = KLu = \lambda K u$. Since zero is not an eigenvalue of the Sturm-Liouville problem, $\lambda \neq 0$, giving $Ku = \lambda^{-1} u$. □

We can now prove the Sturm-Liouville Theorem.

Proof of Theorem 9.1. Since $K : L^2([a, b], \mathbb{R}) \to L^2([a, b], \mathbb{R})$ is compact and self-adjoint, Theorem 8.7 and Corollary 8.8 apply. Since $\mathrm{Range}(K) = \mathcal{D}$ is infinite dimensional, Theorem 8.7 tells us that K has infinitely many non-zero real eigenvalues μ_n tending to zero. From Corollary 8.8, the corresponding eigenfunctions form an orthonormal basis u_n for $L^2([a, b], \mathbb{R})$. By Lemma 9.8, $\lambda_n = \mu_n^{-1}$ are the eigenvalues of the Sturm-Liouville problem and $|\lambda_n| \to +\infty$; furthermore u_n are the eigenfunctions.

Finally, suppose that u is an eigenfunction of the Sturm-Liouville problem and that $\tilde{u}$ is another eigenfunction corresponding to the same eigenvalue. Both satisfy the ODE $(L - \lambda I)v = 0$ with initial condition $v(a) = 0$. But this equation has a unique solution up to multiplication by a scalar. Thus $\tilde{u}$ is a constant multiple of u and the eigenspace in one-dimensional. □

Remark 9.9. The assumption that zero is not an eigenvalue of the Sturm-Liouville problem is not an essential restriction. For any constant c, the eigenvalues of $L + cI$ are $\lambda + c$, where λ are the eigenvalues of L, and the corresponding eigenvectors of the two operators are the same. By adding a suitable constant to q we can always ensure that $\lambda = 0$ is not an eigenvalue of our Sturm-Liouville problem. For example, choose c so that $q + c$ does not change sign in $[a, b]$. For definiteness, assume that $q(x) + c < 0$ for $a \leq x \leq b$. Let u be the (unique) solution of

$$(L_c u)(x) := \frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + (q(x) + c)u(x) = 0, \qquad u(a) = 0,\ u'(a) = 1.$$

Any solution of $L_c v = 0$, $v(a) = 0$, is a constant multiple of u so to show that zero is not an value of the the Sturm-Liouville problem determined by L_c we must show that $u(b) \neq 0$.

Since $u'(a) > 0$ and $u(a) = 0$, it follows that $u(x) > 0$ for $a < x < a + \delta$, for some $\delta > 0$. If $u(x)$ remains strictly positive for all x we are finished. If not, let y be the smallest zero of u such that $y > a$. Then u' must be zero between a and y. If $y \leq b$ then $(pu')' = -(q + c)u$ is positive in (a, y) and so pu' is increasing in (a, y). But pu' is strictly positive at a and so must be strictly positive in (a, y). This contradicts u' vanishing between a and y. Hence $y > b$, i.e. $u(b) \neq 0$. Thus, zero is not an eigenvalue as required.