Introduction to Topology Exercises 2 October 2016

Section A exercises are the easiest; Section B are to be handed in; Section C are harder. Exam-type questions are indicated with an asterisk. To be handed in by 2pm on Wednesday of week 3.

Section A

1. Let G be a group and H a subgroup. The set of left cosets of H in G is denoted by G/H. This can also be described (as a set) as $G/\sim$, where $\sim$ is a certain equivalence relation. What is the equivalence relation?

2. Let $\exp : \mathbb{R} \to S^1$ be the map $\exp(t) = e^{2\pi it} = (\cos 2\pi t, \sin 2\pi t)$. Show that exp is a group homomorphism. What is its kernel? Is the equivalence relation induced by exp (as in Q5 in Exercises 1) the same as the equivalence relation described in Q1 above, with $G = \mathbb{R}$, $H = \ker \exp$?

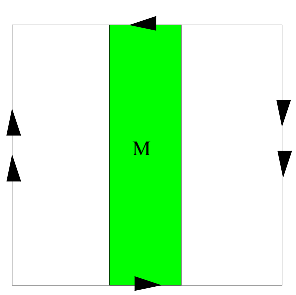
3. Let $q : S^2 \to \mathbb{RP}^2$ be the quotient map. Show that q is an open map: it maps open sets in S^2 to open sets in $\mathbb{RP}^2$.

4. Is the quotient map from the square (thought of as a subspace of $\mathbb{R}^2$) to the torus an open map? Recall: a set is open in the square if it is the intersection of the square with an open set in $\mathbb{R}^2$.

Section B

5. Is the quotient map from the square to $\mathbb{RP}^2$ an open map? [3 marks]

6. The following diagram shows that $\mathbb{RP}^2$ contains a Möbius strip M.



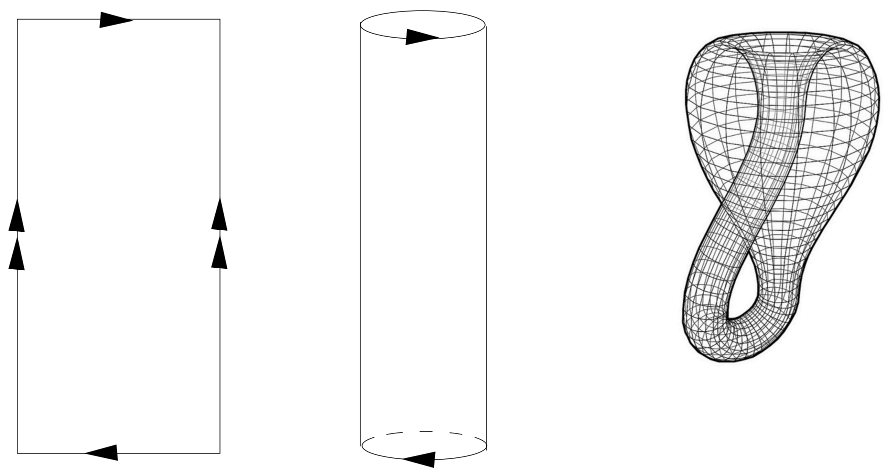
Use the diagram, taking account of the glueings, to describe the complement of M in $\mathbb{RP}^2$. You are allowed to cut it, provided you then glue it back together. Complete the following sentence in as clear a way as possible: "$\mathbb{RP}^2$ is obtained from a Möbius strip by ......" [6 marks]

7. Both the Klein bottle K and $\mathbb{RP}^2$ are 2-dimensional manifolds. This is not obvious from their descriptions as quotients of the square by an equivalence relation. In fact each point x does have an open neighbourhood homeomorphic to an open set in $\mathbb{R}^2$. Show by carefully labelled drawings that this is true if

1. x is in the image of the interior of the square.
2. x in the image of an edge of the square, but not of a vertex.
3. x is the image of a vertex.

Your drawings for (ii) and (iii) have to make use of the fact that the edges of the square are glued together in passing to the quotient. [6 marks]

8. Here is a proof that the Klein bottle can be embedded in $\mathbb{R}^4$. The drawing on the right below shows the image of a failed embedding of the Klein bottle in $\mathbb{R}^3$, $f : K \to \mathbb{R}^3$. The surface has to pass through itself in order for the two ends of the cylinder to glue together as required.



To embed K in $\mathbb{R}^4$, the map f needs one further component, f_4. You can describe f_4 by specifying its value at each point of K. By taking care that f_4 distinguishes points on K which have the same image under f in $\mathbb{R}^3$ (i.e. in the right hand picture), you obtain an embedding. Copy the right hand picture and write suitable values of f_4 on your copy. Your f_4 should of course be continuous. [5 marks]

Section C

9. Recall that the projective plane is the quotient of S^2 by the equivalence relation identifying antipodal points. Show that the map $S^2 \to \mathbb{R}^4$ defined by

$$f(x, y, z) = (yz, xz, xy, y^2 - z^2)$$

passes to the quotient to define an embedding $\mathbb{RP}^2 \to \mathbb{R}^4$ (that is, a homeomorphism onto its image). Suggestion: you have to show that antipodal points have the same image (this is obvious) and that if two distinct points have the same image then they are antipodal. If $f(x, y, z) = f(x', y', z')$ then

$$yz = y'z', \quad xz = x'z', \quad xy = x'y', \quad y^2 - z^2 = y'^2 - z'^2.$$

Multiply together the first two of these equations to get $xyz^2 = x'y'z'^2$. As $xy = x'y'$ then provided $xy \neq 0$, we deduce $z^2 = z'^2$, so $z = \pm z'$. Argue case by case when $xy = 0$. Recall that $x^2 + y^2 + z^2 = x'^2 + y'^2 + z'^2 = 1$.

You may find a cleaner argument than the one suggested here, which is a bit of a mess. Let me know if you do!

10.* A diagram analogous to the one in Q6 above shows that the Klein bottle K contains a Möbius strip M. What is the complement of M in K?