Define a sequence $f_n \in C([0,1],\mathbb{R})$, $n \geq 2$, by $f_n(x) = 0$ for $0 \leq x \leq 1/2 - 1/n$, $f_n(x) = 1$ for $1/2 + 1/n \leq x \leq 1$ and linear in between.

If $m > n$ then

$$\|f_n - f_m\|_1 = \int_0^1 |f_n(x) - f_m(x)|\, dx \leq \frac{2}{n}$$

(since the integral is less than the area of the rectangle with base $[1/2 - 1/n, 1/2 + 1/n]$ and height 1) so $f_n$ is $\|\cdot\|_1$-Cauchy.

We will now argue by contradiction. Suppose $C([0,1],\mathbb{R})$ is $\|\cdot\|_1$-complete. Then there exists $g \in C([0,1],\mathbb{R})$ such that $\|f_n - g\|_1 \to 0$, as $n \to \infty$.

Claim 1: $g(x) = 0$ for all $0 < x < 1/2$.

Justification: Suppose not. Then there exists $0 < x_0 < 1/2$ such that $|g(x_0)| = a > 0$. Since $g$ is continuous, there exists $\delta > 0$ such that $|g(x)| > a/2$ for all $x_0 - \delta < x < x_0 + \delta$. We can choose $\delta$ so that $x_0 + \delta < 1/2$.

Now choose $n_0$ so that $1/2 - 1/n > x_0 + \delta$ for all $n \geq n_0$. Then, for $n \geq n_0$,

$$\|f_n - g\|_1 \geq \int_{x_0-\delta}^{x_0+\delta} |f_n(x) - g(x)|\, dx = \int_{x_0-\delta}^{x_0+\delta} |g(x)|\, dx \geq \delta a > 0,$$

which contradicts $\|f_n - g\|_1 \to 0$, as $n \to \infty$. Therefore the claim holds.

Claim 2: $g(x) = 1$ for all $1/2 < x < 1$.

Justification: Similar to Claim 1.

Combining the claims, $g$ cannot be continuous at $1/2$. So we have a contradiction and so completeness fails to hold.