A proof that $C_{\mathbb{F}}[0,1]$ is not complete with respect to the 2-norm

Harold from https://unifiedmathematics.com

Let the 2-norm on $C_{\mathbb{F}}[0,1]$ be as given. We show that $(C_{\mathbb{F}}[0,1], \|\cdot\|_2)$ is not a Banach space. For $0 < \epsilon < 1/2$, define a map $f_\epsilon : [0,1] \to \mathbb{F}$ by

$$x \mapsto \begin{cases} 0 & \text{if } 0 \leq x < \frac{1}{2} - \epsilon \\ \frac{1}{2\epsilon}(x - (\frac{1}{2} - \epsilon)) & \text{if } x \in \left[\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon\right] \\ 1 & \text{if } 1 \geq x > \frac{1}{2} + \epsilon. \end{cases}$$

Then f is piecewise linear and continuous, as $f(\frac{1}{2} - \epsilon) = 0$ and $f(\frac{1}{2} + \epsilon) = 1$.

Define the map $\iota : C_{\mathbb{F}}[0,1] \to L^2_{\mathbb{F}}[0,1]$ by sending a function $f \in C_{\mathbb{F}}[0,1]$ to its equivalence class in $L^2_{\mathbb{F}}[0,1]$. This is clearly a linear map.

Claim. *ι is an isometry with respect to the 2-norm.*

Proof. Observe that the norm on $L^2_{\mathbb{F}}[0,1]$ is defined by taking a representative of the corresponding equivalence class and computing the same integral as in the definition of the 2-norm on $C_{\mathbb{F}}[0,1]$; i.e., $\|[\iota(f)]\|_2 = \|f\|_2$ for $f \in C_{\mathbb{F}}[0,1]$, where the first norm is in $L^2[0,1]$ and the second in $C[0,1]$. $\square$

Claim. *$\iota(f_\epsilon) \to [1_{[1/2,1]}]$ as $\epsilon \to 0$ in $L^2_{\mathbb{F}}[0,1]$, where $[1_{[1/2,1]}]$ denotes the equivalence class of the characteristic function of the interval $[1/2,1]$ in $L^2_{\mathbb{F}}[0,1]$.*

Proof. Let $0 < \epsilon < 1/2$; then we have

$$\begin{aligned} \|\iota(f_\epsilon) - [1_{[1/2,1]}]\|_2^2 &= \int_0^1 |f_\epsilon(x) - 1_{[1/2,1]}(x)|^2 \,\mathrm{d}x \\ &= \int_0^{1/2} |f_\epsilon(x)|^2 \,\mathrm{d}x + \int_{1/2}^1 |f_\epsilon(x) - 1|^2 \,\mathrm{d}x \\ &= \int_0^\epsilon \left|\frac{x}{2\epsilon}\right|^2 \,\mathrm{d}x + \int_{1/2}^{1/2+\epsilon} |1 - f_\epsilon(1-x) - 1|^2 \,\mathrm{d}x \\ &= \frac{1}{4\epsilon^2} \cdot \frac{\epsilon^3}{3} + \int_{1/2}^{1/2+\epsilon} |f_\epsilon(1-x)|^2 \,\mathrm{d}x = \frac{\epsilon}{12} + \int_0^\epsilon |f_\epsilon(1/2 - x)|^2 \,\mathrm{d}x \\ &= \frac{\epsilon}{12} + \int_0^\epsilon |\frac{1}{2\epsilon}(\frac{1}{2} - x - (\frac{1}{2} - \epsilon))|^2 \,\mathrm{d}x = \frac{\epsilon}{12} + \int_0^\epsilon |\frac{1}{2\epsilon}(x - \epsilon))|^2 \,\mathrm{d}x \\ &= \frac{\epsilon}{12} + \frac{1}{4\epsilon^2}\frac{\epsilon^3}{3} = \frac{\epsilon}{6}. \end{aligned}$$

Thus $\lim_{\epsilon\to 0} \|\iota(f_\epsilon) - [1_{[1/2,1]}]\|_2 = \lim_{\epsilon\to 0} \sqrt{\epsilon/6} = 0$ and $\iota(f_\epsilon) \to [1_{[1/2,1]}]$ as $\epsilon \to 0$, in $L^2_{\mathbb{F}}[0,1]$. $\square$

Claim. *$[1_{[1/2,1]}]$ is not in the range of ι, i.e., $1_{[1/2,1]}$ is not λ-almost everywhere equal to a continuous function, where λ denotes the Lebesgue measure on $[0,1]$.*

Proof. Suppose $1_{[1/2,1]} = f$ λ-a.e. for $f \in C_{\mathbb{F}}[0,1]$. Then every open neighbourhood of $1/2$ contains x, y such that $f(x) = 0$ and $f(y) = 1$, because open neighbourhoods have measure greater than 0. Thus f cannot be continuous in $1/2$, and we have reached a contradiction. $\square$

Claim. *$C_{\mathbb{F}}[0,1]$ is not complete with respect to the 2-norm.*

Proof. Suppose that $C_{\mathbb{F}}[0,1]$ were complete. Then $\iota(C_{\mathbb{F}}[0,1])$ would be a complete subspace of $L^2_{\mathbb{F}}[0,1]$ as ι is an isometry. But $\iota(f_\epsilon) \in \iota(C_{\mathbb{F}}[0,1])$ for all $0 < \epsilon < 1/2$, and $\iota(f_\epsilon) \to [1_{[1/2,1]}]$ as $\epsilon \to 0$. Since $1_{[1/2,1]}$ is not λ-a.e. equal to a continuous function, $[1_{[1/2,1]}] \notin \iota(C_{\mathbb{F}}[0,1])$, and $\iota(C_{\mathbb{F}}[0,1])$ is not a closed subspace of $L^2_{\mathbb{F}}[0,1]$, and cannot be complete. Thus we have reached a contradiction, and $C_{\mathbb{F}}[0,1]$ is not complete. □

Thus, $(C_{\mathbb{F}}[0,1], \|\cdot\|_2)$ is not Banach.

As a corollary, we show that the 2-norm is not equivalent to the standard norm on $C_{\mathbb{F}}[0,1]$.

Suppose that X is a vector space with equivalent two norms, $\|\cdot\|_1$ and $\|\cdot\|_2$. We show that X is complete w.r.t. $\|\cdot\|_1$ if and only if it is complete w.r.t $\|\cdot\|_2$.

Suppose X is complete w.r.t $\|\cdot\|_1$. Let $c, C > 0$ be constants such that $c\|v\|_1 \leq \|v\|_2 \leq C\|v\|_1$ for all $v \in X$, and let $\{x_n\}$ be a sequence in X that is Cauchy w.r.t. $\|\cdot\|_2$. Let $\epsilon > 0$; then there exists N such that $\|x_n - x_m\|_2 < c\epsilon$ for all $n, m \geq N$. Then $\|x_n - x_m\|_1 \leq \|x_n - x_m\|_2/c < \epsilon$ for all $n, m \geq N$. Thus $\{x_n\}$ is Cauchy w.r.t $\|\cdot\|_1$ and by completeness, there exists $x \in X$ such that $\|x_n - x\|_1 \to 0$ for $n \to \infty$. But then $\|x_n - x\|_2 \leq C\|x_n - x\|_1 \to 0$ for $n \to \infty$, and x_n converges to x w.r.t $\|\cdot\|_2$, so X is complete w.r.t $\|\cdot\|_2$.

By symmetry of equivalence of norms and the symmetry of the argument, we conclude that X is complete w.r.t. $\|\cdot\|_1$ iff it is complete w.r.t. $\|\cdot\|_2$.

We have shown that $(C_{\mathbb{F}}[0,1], \|\cdot\|_2)$ is not complete, and we know that $(C_{\mathbb{F}}[0,1], \|\cdot\|_\infty)$ is complete, so $\|\cdot\|_2$ and $\|\cdot\|_\infty$ cannot be equivalent.