MA377 Assignment I

due: Friday 28th October, 2pm, drop-off box outside the Undergraduate Office

Two of the following problems will be marked

Problem 1. (a) Let R be a ring. For $a \in R$ define a ring homomorphism $\varphi_a : R[T] \to R : P(T) \mapsto P(a)$ as the evaluation at a. By restriction of scalars, every φ_a gives the target R the structure of an $R[T]$-module, which we will denote R_a. Show that for $a, b \in R$, there is an $R[T]$-module isomorphism between R_a and R_b if and only if $a = b$.

(b) Let M be an R-module. Show that there is a surjection from a free R-module onto M.

(c) Show that the $\mathbb{Z}$-module $\mathbb{Q}$ is not free.

Problem 2. (a) Compute the homology groups at $\mathbb{Q}^5$ and $\mathbb{Z}^5$ of the complexes

$$\mathbb{Q}^3 \xrightarrow{\begin{pmatrix} 0&1&1\\0&1&1\\2&1&0\\0&1&1\\0&1&1 \end{pmatrix}} \mathbb{Q}^5 \xrightarrow{\begin{pmatrix} 1&1&0&-1&-1\\2&2&0&-2&-2 \end{pmatrix}} \mathbb{Q}^2 \quad \text{and} \quad \mathbb{Z}^3 \xrightarrow{\begin{pmatrix} 0&1&1\\0&1&1\\2&1&0\\0&1&1\\0&1&1 \end{pmatrix}} \mathbb{Z}^5 \xrightarrow{\begin{pmatrix} 1&1&0&-1&-1\\2&2&0&-2&-2 \end{pmatrix}} \mathbb{Z}^2.$$

(b) Let

$$\begin{array}{ccccccccc} M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & M_4 & \longrightarrow & M_5 \\ {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_3}\downarrow & & {\scriptstyle f_4}\downarrow & & {\scriptstyle f_5}\downarrow \\ N_1 & \longrightarrow & N_2 & \longrightarrow & N_3 & \longrightarrow & N_4 & \longrightarrow & N_5 \end{array}$$

be a commutative diagram of R-modules in which the rows are exact sequences. Show the Five-Lemma: If f_1, f_2, f_4 and f_5 are isomorphisms then so is f_3.

Problem 3. Let k be a field, and let G be a group.

(a) A *representation of* G is a k-vector space V together with a map $G \times V \to V : (g, v) \mapsto gv$ such that i) $\forall g \in G$ the map $V \to V : v \mapsto gv$ is k-linear, ii) $\forall g, h \in G, v \in V : g(hv) = (gh)v$, and iii) $\forall v \in V : 1 \cdot v = v$. A homomorphism of G representations is a k-linear map $\varphi : V \to W$ such that $\varphi(gv) = g \cdot \varphi(v)$ for all $g \in G$ and $v \in V$.

Show that every $k[G]$-module M is a G-representation via the map $G \times M \to M : (g, v) \mapsto \langle g \rangle \cdot v$, and every $k[G]$-module homomorphism is a homomorphism of G-representations. Conversely, show that every G-representation has a unique $k[G]$-module structure, and every homomorphism of G-representations is a $k[G]$-module homomorphism.

(b) Let G be a finitely generated abelian group. Use the structure theorem for such groups and the Isomorphism Theorem for rings to give an explicit ring isomorphism between $k[G]$ and a quotient of a polynomial ring (in possibly several variables) with coefficients in k.

Problem 4. Let R be a ring. Compute the center of $M_n(R)$ and of $R[T]$ in terms of the center of R.