Measure Theory MA359

Assignment 1

Due to be submitted before 2 pm on Tuesday, October 25, to the drop off box in front of the undergraduate office

A. Warm up questions

(Not for credit. Not to be submittted.)

(A1) Let $X = \{0, 1, 2\}$ and $\mathcal{C} = \{\{0\}\}$. Enumerate the class of all σ-algebras containing $\mathcal{C}$ and give $\sigma(\mathcal{C})$.

(A2) Let $X = \{m, \alpha, x, d\}$ and $\mathcal{C} = \{\{m, \alpha, x\}, \{\alpha, x\}\}$. What is the σ-algebra $\sigma(\mathcal{C})$?

(A3) Let $f : \mathbb{R} \to \mathbb{R}$ be a monotone function. Show that f is Borel measurable.

(A4) Suppose $\mathcal{C} \subset \mathcal{P}(X)$ is nonempty. Let $\mathcal{A}(\mathcal{C})$ be the minimal algebra containing $\mathcal{C}$. Show that $\mathcal{A}(\mathcal{C})$ consists of sets of the form

$$\cup_{i=1}^{m} \cap_{j=1}^{n_i} A_{i,j},$$

where for each i, j either $A_{i,j} \in \mathcal{C}$ or $A_{i,j}^{c} \in \mathcal{C}$ and where the m sets $\cap_{j=1}^{n_i} A_{i,j}$, $1 \leq i \leq m$, are mutually disjoint.

Notice that we can explicitly represent the sets from $\mathcal{A}(\mathcal{C})$, even though it is in general not possible for the sets from the σ-algebra $\sigma(\mathcal{C})$.

(A5) Prove that a finitely additive measure μ is a measure if and only if it is upward monotone convergent.

Here, finitely additive measure on $(X, \mathcal{A})$ is a map $\mu : \mathcal{A} \to [0, \infty]$ such that $\mu(\emptyset) = 0$ and $\mu(\cup_{n=1}^{k} A_n) = \sum_{n=1}^{k} \mu(A_n)$ for any disjoint $A_1, \ldots, A_k \in \mathcal{A}$ and it is upward monotone convergent if $\mu(\cup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$ whenever $A_1, A_2, \dots \in \mathcal{A}$ and $A_1 \subset A_2 \subset \dots$.

(A6) Suppose $(X, \mathcal{A}, \mu)$ is a measure space and A_n is a sequence of sets in $\mathcal{A}$. We define the sets:

$$B = \{x \in X : x \in A_n \text{ for infinitely many } n\},$$
$$C = \{x \in X : \exists\, n_0(x) \text{ such that } x \in A_n\, \forall n \geq n_0(x)\}.$$

(a) Check that

$$B = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k \text{ and } C = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k$$

and conclude that B, C belong to $\mathcal{A}$.

(b) Show that

$$\mathbb{1}_B = \limsup_{n\to\infty} \mathbb{1}_{A_n},$$
$$\mathbb{1}_C = \liminf_{n\to\infty} \mathbb{1}_{A_n}.$$

(Remark: In view of the above it is natural to introduce the notation $B = \limsup_{n\to\infty} A_n$ and $C = \liminf_{n\to\infty} A_n$.)

(c) Show that $\mu(C) \leq \liminf_{n\to\infty} \mu(A_n)$ and if $\mu(\bigcup_{n=1}^{\infty} A_n) < \infty$, then

$$\limsup_{n\to\infty} \mu(A_n) \leq \mu(B).$$

(d) Prove the **Borel - Cantelli** Lemma, i.e.,

if $\sum_{n=1}^{\infty} \mu(A_n) < \infty$, *then* $\mu(B) = \mu(\limsup_{n\to\infty} A_n) = 0$.

B. Questions for credit

(Due before 2 pm on Tuesday, October 25)

(B1)

(a) Suppose that $\mathcal{A}_n$ are algebras satisfying $\mathcal{A}_n \subset \mathcal{A}_{n+1}$. Show that $\cup_{n=1}^{\infty} \mathcal{A}_n$ is an algebra.

(b) Check that if all $\mathcal{A}_n$ above are σ-algebras, their union need not be a σ-algebra.

Is a countable union of σ algebras (whether monotone or not) an algebra? *Hint:* Try considering the set of all positive integers with its σ-algebras $\mathcal{A}_n = \sigma(\mathcal{C}_n)$ where $\mathcal{C}_n = \mathcal{P}(\{1, 2, \ldots, n\})$.

Check that if $\mathcal{B}_1$ and $\mathcal{B}_2$ are σ-algebras, their union $\mathcal{B}_1$ and $\mathcal{B}_2$, need not be an algebra.

(B2) Show that $\mathcal{B}(\mathbb{R})$ is countably generated; that is, show the Borel sets are generated by a countable class $\mathcal{A}$.

(B3) Show that the periodic sets of $\mathbb{R}$ form a σ-algebra; that is, let $\mathcal{B}$ be the class of sets A with the property that $x \in A$ implies $x \pm n \in A$ for all natural numbers n. The $\mathcal{B}$ is a σ-algebra.

(B4) Let $\overline{R} = [-\infty, \infty]$ be extended real line. The Borel σ-algebra $\mathcal{B}(\overline{\mathbb{R}})$ is generated by the sets $[-\infty, x), x \in \mathbb{R}$.

Think of $\overline{R} = [-\infty, \infty]$ as homeomorphic in the topological sense to $[-1, 1]$ under the transformation

$$x \mapsto \frac{x}{1 - |x|}$$

from $[-1, 1]$ to $[-\infty, \infty]$. Consider the usual topology on $[-1, 1]$ and map it into a topology on $[-\infty, \infty]$. This defines a collection of open sets on $[-\infty, \infty]$ and these open sets can be used to generate a Borel σ-algebra. How does this σ-algebra compare with the standard $\mathcal{B}(\overline{\mathbb{R}})$? Provide needed arguments!