Introduction to Topology   Exercises 4   October 2016

To be handed in by 2pm on Wednesday of week 5. By all means ask, by e-mail, if any of the questions is not clear.

Heartfelt request from the TAs: Please

1. **Write your name on your solutions**
2. **Don't write pages and pages. All the questions have quite brief solutions.**

Section A

1. Recall that if $\sim$ is the equivalence relation on $I = [0,1]$ which identifies the end points, then $I/\sim$ is homeomorphic to S^1. In the following exercise it may be useful to fix a homeomorphism φ from $I/\sim$ to S^1, sending the class of the end-points to the point $1 \in S^1$.

Show that
(i) there is a natural bijection

$$\text{Loops}(X, x_0) \to \text{Maps}\left((S^1, 1) \to (X, x_0)\right)$$

(where "Maps" means "continuous maps".
(ii) two loops f, g in X based at x_0 are path-homotopic if and only if the corresponding maps $(S^1, (1,0)) \to (X, x_0)$ are homotopic through maps $f_t : S^1 \to X$ all sending $(1,0)$ to x_0.

2. A space is *contractible* if it is homotopy-equivalent to a point. Show that D^2 is contractible but S^1 is not.

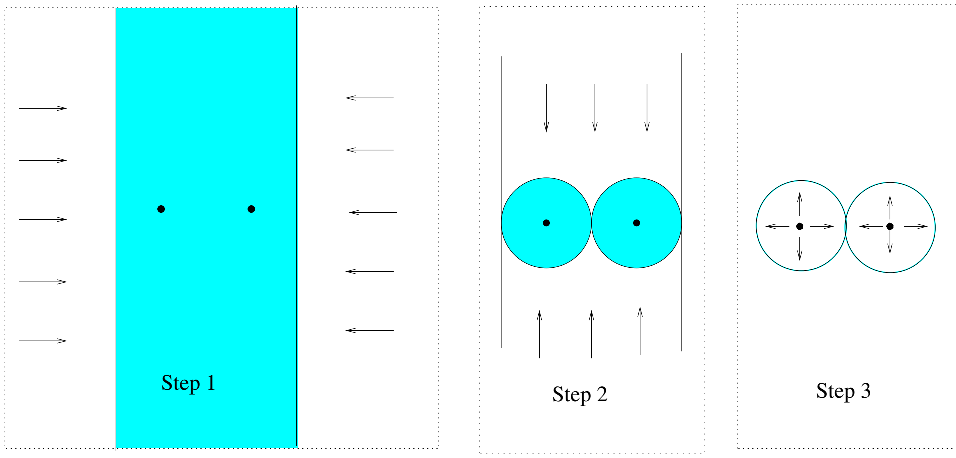
3. Show that homotopy equivalence of spaces is an equivalence relation.

4. Suppose that $B \subset A \subset X$. Show that

1. if B is a retract of A and A is a retract of X then B is a retract of X
2. if B is a deformation retract of A and A is a deformation retract of X then B is a deformation retract of X.

5. Show that

1. S^{n-1} is a deformation retract of $\mathbb{R}^n \setminus \{0\}$.
2. The figure 8 (two circles with a single point in common) is a deformation retract of $\mathbb{R}^2 \setminus \{x_0, x_1\}$. Hint: Do the retraction in stages, as suggested by Q 4.2 and the following picture, in which the arrows indicate the direction in which the points outside the shaded region must slide during a deformation retraction to the shaded region.



Section B

6.(a) Let ∂I^2 be the boundary of the square I^2, and let P be an interior point of I^2. Find a retraction $r : (I^2 \setminus P) \to \partial I^2$.

(b) On page 18 of the Lecture Notes posted on the module homepage, five different quotients of the square are indicated. Denote them by $X_1, \ldots, X_5$ (so X_1 is a cylinder, etc). Show that in each case the retraction $(I^2 \setminus P) \to \partial I^2$ passes to the quotient to define a retraction $X_i \setminus [P] \to (\partial I^2)/\sim$. Here in each case $(\partial I^2)/\sim$ means the image of ∂I^2 in X_i, and $[P]$ means the image of P in X_i.

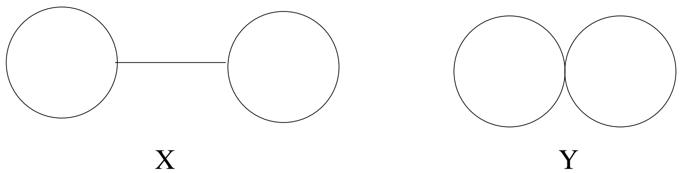
(c) In each of the five cases in (b), draw (a space homeomorphic to) $(\partial I^2)/\sim$. [5 marks]

7. A *graph* is the space obtained by taking a collection of isolated points (the "nodes") and gluing the ends of a collection of disjoint copies of the interval $I = [0, 1]$ to some of them. For example, each of the spaces $(\partial I^2)/\sim$ in 6(c) is a graph. In each of the following cases, find a graph G contained in the space X, such that G is a deformation-retract of X. In all cases, $P_1, \ldots, P_k$ are distinct points. Give a very brief argument for each case, perhaps by means of drawings as in Q5.2.

1. $X = \mathbb{R}^2 \setminus \{P_1, \ldots, P_k\}$
2. $X = T \setminus \{P_1, P_2\}$ (where T is the torus).
3. $X = \mathbb{RP}^2 \setminus \{P_1, P_2\}$.

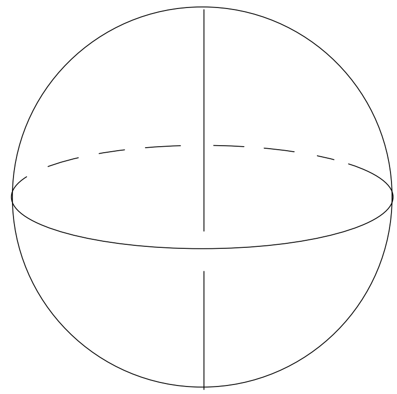
[5 marks]

8. Are the two graphs X and Y shown here homotopy equivalent?



[5 marks]

9. Let S^1 be the circle embedded in the usual way in $\mathbb{R}^2 \subset \mathbb{R}^3$, and let X be the space shown below, consisting of a (hollow) sphere together with one of its diameters. Then $\mathbb{R}^3 \smallsetminus S^1$ has X (suitably situated in $\mathbb{R}^3 \smallsetminus S^1$) as a deformation retract. *To do:* Show this by drawing X in $\mathbb{R}^3 \smallsetminus S^1$ and indicating by means of arrows how the deformation retraction takes place.



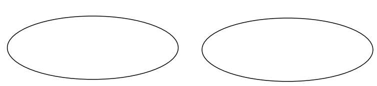
[5 marks]

Section C

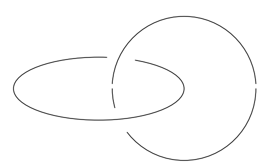
10.* Do Q 6 with "retraction" replaced by "deformation-retraction".

11.* What is $\pi_1(\mathbb{R}^3 \smallsetminus S^1, x_0)$? A partially rigorous answer is better than none.

12. Let A_1 and A_2 be the two subspaces of $\mathbb{R}^3$ shown below. Find spaces X_1 and X_2, analogous to the space X of Q 9, such that $\mathbb{R}^3 \smallsetminus A_1$ has X_1 as deformation retract and $\mathbb{R}^3 \smallsetminus A_2$ has X_2 as deformation retract. See Hatcher pp. 46-47 for answers.



A1



A2

13. (Hatcher, page 38) Let $[S^1, X]$ be the set of homotopy classes of map $S^1 \to X$, with no condition on basepoints. By Q1, we can regard $\pi_1(S^1, (1,0))$ as the set of basepoint-preserving-homotopy classes of maps $(S^1, (1,0)) \to (X, x_0)$. Then there is an obvious map $\theta : \pi_1(S^1, (1,0)) \to [S^1, X]$, obtained by ignoring basepoints.

1. Show that θ is surjective.
2. Show that if $[f], [g] \in \pi_1(X, x_0)$ then $\theta([f][g][\bar{f}] = \theta([g])$. By now I hope it is clear that you should make some drawings to see why this is true.
3. Show that if $\theta([f_1]) = \theta([f_2])$ then $[f_1]$ is conjugate to $[f_2]$.
4. Could there be a group structure on $[S^1, X]$ with respect to which θ is a homomorphism?